이학박사 학위논문

Regularity results for generalized elliptic problems in bounded domains

(유계영역에서 정의된 일반화된 타원형 방정식의 정칙성)

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이 논문을 이학박사 학위논문으로 제출함

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Regularity results for generalized elliptic problems in bounded domains

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Abstract

Three different types of problems will be studied in this thesis. The three problems are the G-Laplace equation in a convex domain, a quasilinear equation with p-growth condition in a quasiconvex domain and generalized steady Stokes system in a Reifenberg flat domain. In each problem, we focus on a gradient estimate of a weak solution.

At first, we prove local boundedness of of the gradient for the homogeneous G-Laplace equation in a convex domain under vanishing Neumann boundary condition. G is a Young function which is a non-decreasing convex function such that $G(0) = 0$ and $\lim_{t\to+\infty} \frac{G(t)}{t} = +\infty$. In this problem, one of our interests is a convex domain, since Lipschitz regularity of a solution to even the Laplace equation cannot be obtained in a Lipschitz domain.

Next, we derive Calderón-Zygmund type estimate for the solution to quasilinear equation with p-growth condition in a quasiconvex domain, which is locally trapped by two convex domains. As far as the domain is concerned, our regularity assumption on the boundary is weaker than any other one reported in this direction. In addition, we extend our result in Lebesgue spaces to Orlicz spaces.

In last chapter, We prove the global weighted L^q-estimates for the gradient of the weak solution and an associated pressure under the assumptions that the coefficients have small BMO (bounded mean oscillation) semi-norms and the domain is sufficiently flat in the Reifenberg sense. On the other hand, a given weight is assumed to belong to a Muckenhoupt class. Our result generalizes the global $W^{1,q}$ estimate for a solution with respect to the Lebesgue measure for the Stokes system in a Lipschitz domain.

Key words: Quasilinear elliptic equation, Lipschitz regularity, Global gradient estimate, Stokes equation, Irregular domain.
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Contents

Chapter 1

Introduction

In this thesis, we are investigating on the uniform estimates for the gradient of solutions to the three different kinds of elliptic boundary value problems on each different bounded domain. The three problems are as follows:

The problem for Chapter 2 is a generalized p-Laplace equation, which is the so called G-Laplace equation

$$\begin{cases} \operatorname{div}\left(\frac{g(|Dv|)}{|Dv|}Dv\right) &= 0 \ \text{ in } \quad \Omega \cap B_{4r}(x_0) \\ \frac{g(|Dv|)}{|Dv|}D_\nu v &= 0 \ \text{ on } \quad \partial\Omega \cap B_{4r}(x_0). \end{cases}$$

Here Ω is a bounded convex domain in $\mathbb{R}^n$, $r > 0$, $x_0 \in \partial\Omega$ and ν is the unit outward normal to $\partial\Omega$, while G is a Young function and $g(t) = G'(t)$.

In Chapter 3, it is a nonlinear elliptic equation with p-growth condition

$$\begin{cases} \operatorname{div} \mathbf{a}(x, Du) = \operatorname{div}(|\mathbf{f}|^{p-2}\mathbf{f}) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Here Ω is a bounded (δ, σ, R)-quasiconvex domain in $\mathbb{R}^n$ and $1 < p < \infty$, while the vector field $\mathbf{a}(x, \xi)$ satisfies ellipticity, p-growth and (δ, R)-vanishing conditions.

In Chapter 4, it is a generalized incompressible steady Stokes system

$$\begin{cases} \operatorname{div}\,(A(x)\nabla u) - \nabla p &= \operatorname{div}\,\mathbf{F} \quad \text{in} \quad \Omega \\ \operatorname{div}\, u &= 0 \qquad\ \ \text{in} \quad \Omega \\ u &= 0 \qquad\ \ \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is a bounded (δ, R)-Reifenberg flat domain in $\mathbb{R}^n$ and $A(x)$ satisfies uniform ellipticity, boundedness and (δ, R)-vanishing condition. The definition for each (δ, σ, R)-quasiconvex domain, (δ, R)-Reifenberg flat domain and (δ, R)-vanishing condition is written in each chapter. One of the elementary issues of the regularity theory is an impact of the right hand side of equations, i.e. how a solution can be more regular by the regularity of terms in the right hand side. The classical Calderón Zygmund theory for

$$-\Delta u = -\text{div } Du = -\text{div } |f|^{p-2} f$$

gives the following regularity.

i) $f \in L^q(\mathbb{R}^n)$ implies $Du \in L^q(\mathbb{R}^n)$ for $1 < q < \infty$

ii) $f \in C^{k,\alpha}(\mathbb{R}^n)$ implies $Du \in C^{k,\alpha}(\mathbb{R}^n)$ for $k \in \mathbb{N}$ and $\alpha \in (0,1)$.

From [46], the non-linear Calderón-Zygmund theory for

$$-\text{div } |Du|^{p-2} Du = -\text{div } f$$

gives the same result with i). This result motivates our works which are the gradient estimates for linear and nonlinear problems from the perspective of the regularity of the right hand side.

Another thing is the influence of the regularity of the boundary of a bounded domain on the solution. In [47], Jerison and Kenig showed that the solution to

$$\begin{cases} \Delta u &= \text{div } f \qquad \text{in } \Omega \\ \quad u &= 0 \qquad\quad \text{on } \partial\Omega, \end{cases}$$

satisfies the following gradients estimate

$$\|Du\|_{L^p(\Omega)} \leq c\|f\|_{L^p(\Omega)} \text{ for } 1 < p < \infty, \tag{1.1}$$

where Ω is a C^1-domain. But it does not hold for Lipschitz domains for $n \geq 3$ and $p \geq 3$, since there exits a Lipschitz domain which makes the estimate fails even if $f \in C_0^\infty(\Omega)$. From the counterexample in [47], it could be guessed that this gradient estimate holds though Ω is a Lipschitz domain, if its Lipschitz constant is sufficiently small. This guess was proved correct by Byun and Wang in [13]. They proved that the estimate (1.1) for the following linear elliptic equation

$$\begin{cases} \text{div } (A(x)Du) & = \text{div } \mathbf{F} \quad \text{in} \quad \Omega \\ u & = 0 \quad\quad\ \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded (δ, R)-Reifenberg domain, which contains a Lipschitz domain with Lipschitz constant less than δ (see [70]), and $A(x)$ uniform ellipticity, boundedness and (δ, R)-vanishing condition, which means that it is possibly discontinuous. As proved in [3], the estimate (1.1) for a solution to a linear elliptic equation (1.2), where Ω is a convex domain and $A(x)$ is continuous, is valid, but a convex domain is possibly not a (δ, R)-Reifenberg domain. To cover (δ, R)-Reifenberg flat domains and convex domains, Jia and Wang [48] introduced (δ, σ, R)-quasiconvex domains which include those two kinds of domains, and proved estimate (1.1) for a linear elliptic equation (1.2), where Ω is a (δ, σ, R)-quasiconvex domain and $A(x)$ is under the same conditions as in [13]. From the perspective of equation, this kind of gradient estimate for linear elliptic equation in (δ, R)-Reifenberg flat domain was extended to nonlinear elliptic equation with p-growth condition,

$$\begin{cases} \text{div}\, \mathbf{a}(x, Du) = \text{div}\, (|\mathbf{f}|^{p-2}\mathbf{f}) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

in [18]. From this result, we proved the Calderón-Zygmund type estimates for a nonlinear elliptic equation in a (δ, σ, R)-quasiconvex domain in Chapter 3.

On the other hand, the result of [13] was extended to linear elliptic systems in [17]. Since the steady Stokes system, $\Delta u - \nabla p = \text{div } F$, $\text{div } u = 0$, has similar properties with the Laplace system, $\Delta u = \text{div } F$, we proved a Calderón-Zygmund type estimate for the solution to a steady generalized Stokes system with Dirichlet boundary condition in a (δ, R)-Reifenberg flat domain. Here, generalization means that $\text{div}(A(x)Du)$ is considered as diffusion term instead of $\Delta u = \text{div}(Du)$.

This regularity of a solution to partial differential equation in an irregular domains is obtained by local comparison with a solution to the limiting equation in the locally approximated domain in the Hausdorff distance sense. If it is known that a solution to the limiting equation is sufficiently regular, the regularity of this solution is transferred to a solution of the original equation. Here, each locally approximated domain of a Reifenberg flat domain and a quasiconvex domain is a half ball and a convex domain respectively. As a C^1-boundary can be considered as locally flat boundary by changing coordinates near a point on the boundary, there are many results about regularity in a half ball. Meanwhile, since a convex domain is a Lipschitz domain, theorems which are valid in a Lipschitz domain hold in a convex domain, too. It means that it's meaningful to obtain regularity in a convex domain which is not true in a Lipschitz domain. One of them is Lipschitz regularity of a solution to some partial differential equation, as written in previous paragraph. There are some works about Lipschitz regularity of a solution to some partial differential equation in a convex domain in [4, 22, 23, 24, 58]. According to [22, 23], the solution to the homogeneous quasilinear equation of the form $\mathrm{div}(a(|Du|)Du) = 0$ in a convex domain under both vanishing Dirichlet boundary condition and vanishing Neumann condition should be a constant almost everywhere. So it needs an other approach to obtain local boundary Lipschitz regularity for a solution to a homogeneous problem in a convex domain. Local Lipschitz regularity for the p-Laplace equation in a convex domain was proved in [4]. And we extended it to the G-Laplace equation in chapter 2. We would like to mention that there are some results that the regularity of a solution to the p-Laplace equation was generalized to the regularity of the G-Laplace equation. $C^{1,\alpha}$-regularity for a solution to the p-Laplace equation was proved in [34, 55, 71] and Lieberman [56] generalized the $C^{1,\alpha}$-regularity for a equation with G-growth. $W^{2,2}$-regularity is well-known for the p-Laplace equation (see [40]) and it is valid for a solution to the G-Laplace equation [25].

Chapter 2

Local Lipschitz regularity in convex domains

2.1 Overview

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a convex domain and let x_0 be a point on the boundary $\partial\Omega$. We study a local Lipschitz regularity of weak solutions to a general class of quasilinear equations with Neumann boundary condition,

$$\begin{cases} \operatorname{div}\left(\frac{g(|Dv|)}{|Dv|}Dv\right) = 0 \text{ in } \Omega \cap B_{4r}(x_0) \\ \frac{g(|Dv|)}{|Dv|}D_\nu v = 0 \text{ on } \partial\Omega \cap B_{4r}(x_0), \end{cases} \tag{2.1}$$

where ν is the unit outward normal vector to $\partial\Omega$ and r satisfies $\Omega \cap B_{4r}(x_0) \neq \emptyset$. We recall some notations, definitions and properties concerning a suitable Young function $G(t) = \int_0^t g(s)ds$ from section 2. We start with our assumptions on G, as stated below.

Assumption 2.1.1. Let G be a Young function belonging to $C^1[0,\infty) \cap C^2(0,\infty)$. Moreover, we assume that

$$g(t) \sim tg'(t), \tag{2.2}$$

where $f \sim h$ means that there exist constants c_0, $c_1 > 0$ such that $c_0 f \leq h \leq$

$c_1 f$, and that $g(t)$ satisfies $0 < i_g \le s_g < \infty$, where

$$i_g := \inf_{t>0} \frac{tg'(t)}{g(t)} \text{ and } s_g := \sup_{t>0} \frac{tg'(t)}{g(t)}. \tag{2.3}$$

We note that the following inequality is derived from (2.3) (see [56])

$$\frac{tg(t)}{1+s_g} \le G(t) \le \frac{tg(t)}{1+i_g} \text{ for } t \ge 0. \tag{2.4}$$

We note that the assumptions above imply that both G and G^* satisfie Δ_2-condition and that $\Delta_2(G)$ depends only on the constants in (2.3). Then it gives $G \in \nabla_2$ by [65, Theorem 2.2.3]. In addition, we impose following condition below on G which means that g' is Hölder continuous away from zero.

Assumption 2.1.2. Let G satisfy assumption 4.31, and assume that there exists $\beta \in (0,1]$ and $c > 0$ such that

$$|g'(s+t) - g'(t)| \le cg'(t) \left(\frac{|s|}{t}\right)^{\beta}$$

for all $t > 0$ and $s \in \mathbb{R}$ with $|s| < \frac{1}{2}t$.

In [55], which appeared in 1983, Lewis proved $C^{1,\alpha}$ regularity of a weak solution to p-Laplace equation and it was generalized by Lieberman in [56] for a bounded weak solution to an elliptic equation under the setting of Orlicz spaces. In the case of the system, everywhere $C^{1,\alpha}$-regularity was proved by Diening, Stroffolini and Bianca in [28]. Esposito, Mingione and Trombetti proved the Lipschitz regularity of a weak solution to quasilinear equations and a minimizer of functionals with G-growth in [33]. Later on, Cianchi and Maz'ya proved the global Lipschitz regularity of a weak solution to quasilinear equations of the form $-\text{div}\ (a(|Du|)Du = f$ in convex domains for both Dirichlet and Neumann problems, $u = 0, u_\nu = 0$, respectively on $\partial\Omega$, where $f \in L^{n,1}(\Omega)$, a Lorentz space, in [22]. On the other hand, Baroni proved pointwise gradient estimates via linear Riesz potentials in [5]. We mention the very interesting paper [4] in which Banerjee and Lewis proved the local Lipschitz regularity near the boundary for p-Laplace system both for Dirichlet and Neumann boundary conditions on convex domains. In this paper we extend the result of [4] to a general class of quasilinear equations with Neumann condition. Precisely, we have the following theorem:

Theorem 2.1.3. *Let G satisfy assumption 2.1.2. Then there exists a constant $c = c(G, n, \Omega) > 0$ such that if $v \in W^{1,G}(\Omega)$ is a weak solution to (2.1), then*

$$\|G(|Dv|)\|_{L^\infty(\Omega\cap B_r(x_0))} \leq \frac{c}{r^n}\int_{\Omega\cap B_{4r}(x_0)} G(|Dv|)dx. \tag{2.5}$$

2.2 Elemental definitions and auxiliary results

2.2.1 Young function G and auxiliary results

Definition 2.2.1. Let g be a real function defined on $[0, \infty)$ with following properties:

i) $g(0) = 0$, $g(t) > 0$ for $t > 0$, $\lim\limits_{t\to\infty} g(t) = \infty$;

ii) g is increasing;

iii) g is left continuous.

Then the real function G defined on $[0, \infty)$ as

$$G(t) = \int_0^t g(s)\, ds$$

is called a Young function.

The complementary function G^* is defined by

$$G^*(s) := \sup_{t\geq 0} \left(st - G(t)\right).$$

Definition 2.2.2. Let G be a Young function.

1. G is said to satisfy the Δ_2-condition, denoted by $G \in \Delta_2$, if there exists $k_1 > 0$ such that $G(2t) \leq k_1 G(t)$ for all $t > 0$. We denote the constant k_1 as $\Delta_2(G)$.

2. G is said to satisfy the ∇_2-condition, denoted by $G \in \nabla_2$, if there exists $k_2 > 1$ such that $2k_2 G(t) \leq G(k_2 t)$ for all $t > 0$.

Here we can see that for $G \in \Delta_2$, it holds that

$$\text{if } \lambda \geq 1, \text{ then } G(\lambda t) \leq \lambda^{k_1} G(t) \text{ for all } t \geq 0, \tag{2.6}$$

where $k_1 = \Delta_2(G)$. And for all $\epsilon > 0$ there exists a positive constant $c = c\,(\epsilon, \Delta_2(G))$ such that

$$st \leq \epsilon G^*(s) + cG(t) \text{ for all } s, t \geq 0,$$

which is called Young's inequality. Particularly, $st = G(s) + G^*(t)$ holds when $t = g(s)$, or $s = (G^*)'(t)$.

Definition 2.2.3. Let G be a Young function. Then the Orlicz space $L^G(\Omega)$ is the Banach space of those measurable functions $u : \Omega \to \mathbb{R}$ whose Luxemburg norm

$$\|u\|_{L^G(\Omega)} = \inf \left\{ \lambda : \int_\Omega G\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1 \right\}$$

is finite. And the Orlicz-Sobolev space $W^{1,G}(\Omega)$ is the set of every weakly differentiable function in $L^G(\Omega)$ whose gradient belongs to $L^G(\Omega)$, too. It is equipped with the norm

$$\|u\|_{W^{1,G}(\Omega)} = \|u\|_{L^G(\Omega)} + \|\nabla u\|_{L^G(\Omega)}.$$

We need the following Lemma concerning Orlicz-Sobolev boundary trace embeddings from [21].

Lemma 2.2.4. *Let $\Omega \subset \mathbb{R}^n, n \geq 2$, be a bounded Lipschitz domain, let T be the trace operator, and let G be a Young function satisfying*

$$\int_0^{t_0} \left(\frac{t}{G(t)}\right)^{\frac{1}{n-1}} dt < \infty \tag{2.7}$$

for some $0 < t_0 < \infty$.

1. *Assume that*

$$\int_{t_1}^{\infty} \left(\frac{t}{G(t)}\right)^{\frac{1}{n-1}} dt = \infty \tag{2.8}$$

for some $0 < t_1 < \infty$. Let G_T be the Young function defined by

$$G_T(t) = \int_0^{H^{-1}(t)} \left(\frac{G(\tau)}{\tau}\right)^{\frac{n-2}{n-1}} H(\tau)^{\frac{1}{1-n}} d\tau,$$

where $H(\tau) = \left(\int_0^\tau \left(\frac{s}{G(s)}\right)^{\frac{1}{n-1}} ds\right)^{\frac{n-1}{n}}$ for $\tau \geq 0$. Then there exists a constant $c = c(G,\Omega) > 0$ such that

$$\|Tv\|_{L^{G_T}(\partial\Omega)} \leq c\|v\|_{W^{1,G}(\Omega)}$$

for all $v \in W^{1,G}(\Omega)$.

2. *Assume that*

$$\int_{t_1}^\infty \left(\frac{t}{G(t)}\right)^{\frac{1}{n-1}} dt < \infty \tag{2.9}$$

for some $0 < t_1 < \infty$. Then there exists a constant $c = c(G,\Omega) > 0$ such that

$$\|Tv\|_{L^\infty(\partial\Omega)} \leq c\|v\|_{W^{1,G}(\Omega)}$$

for all $v \in W^{1,G}(\Omega)$.

Definition 2.2.5. Let F and G be young functions. We say that F *dominates G near infinity* if there exist positive constants c and t_0 such that $G(t) \leq F(ct)$ for $t \geq t_0$. Two functions F and G are called *equivalent near infinity* if each dominates the other near infinity. We write $G \preceq F$ to denote that F dominates G near infinity, and $F \approx G$ to denote that F and G are equivalent near infinity.

Lemma 2.2.6. *[2, Theorem 8.12] Let Ω be a bounded domain. The embedding $L^F(\Omega) \to L^G(\Omega)$ holds if and only if $G \preceq F$.*

Corollary 2.2.7. *Under the same assumptions as in lemma 2.2.4, we further assume that G satisfies (2.4). Then $G \preceq G_T$ and the trace operator, $T : W^{1,G}(\Omega) \to L^G(\partial\Omega)$, is a compact operator.*

Proof. We may assume that the inequality (2.7) holds, since $L^G(\Omega)$ is unchanged even though we replace G by an equivalent Young function near infinity satisfying this inequality, if necessary, and hence $W^{1,G}(\Omega)$ is also unchanged, by Lemma 2.2.6.

Using a change of variables, we can see that

$$G_T(t) = \frac{n}{n-1}\int_0^t \frac{G(H^{-1}(s))}{H^{-1}(s)}ds \text{ for } t \geq 0. \tag{2.10}$$

Since $\frac{G(t)}{t}$ is increasing, there is a constant $k > 0$ such that for sufficiently large t,

$$G(t) \geq kt.$$

From this, we see that

$$H(t) \leq c_1 t^{\frac{n-1}{n}}$$

for some constant $c_1 > 0$ and sufficiently large t and then

$$H^{-1}(\tau) \geq c_2 \tau^{\frac{n}{n-1}} \tag{2.11}$$

for some constant $c_2 > 0$ and sufficiently large τ. One can choose $0 < \alpha < i_g$ such that

$$\frac{G(s)}{s^{1+\alpha}} \leq \frac{G(t)}{t^{1+\alpha}} \text{ if } s \leq t, \tag{2.12}$$

as

$$\frac{d}{dt}\left(\frac{G(t)}{t^{1+\alpha}}\right) = \frac{tg(t) - (1+\alpha)G(t)}{t^{2+\alpha}} \geq \frac{(i_g - \alpha)G(t)}{t^{2+\alpha}} > 0,$$

provided $0 < \alpha < i_g$. In view of definition 2.2.5 and the identity (2.10), we observe that the followings are equivalent:

1. $G \preceq G_T$;
2. $\lim\limits_{t\to\infty} \dfrac{G_T(t)}{G(\lambda t)} = \infty$ for all $\lambda > 0$;
3. $\lim\limits_{t\to\infty} \dfrac{\int_0^t \frac{G(H^{-1}(s))}{H^{-1}(s)}ds}{G(\lambda t)} = \infty$ for all $\lambda > 0$.

We now claim that the above limit (3) holds. Indeed, by l'Hôpital's rule, (3) follows from showing that

$$\lim_{t\to\infty} \frac{\frac{G(H^{-1}(t))}{H^{-1}(t)}}{g(\lambda t)} = \infty \text{ for all } \lambda > 0,$$

if and only if

$$\lim_{t\to\infty} \frac{G\left(H^{-1}(t)\right)}{H^{-1}(t)} \frac{t}{G(\lambda t)} = \infty \text{ for all } \lambda > 0, \tag{2.13}$$

from the fact that $\frac{G(t)}{t} \leq g(t) \leq (1+s_g)\frac{G(2t)}{t}$.

Combining (2.11) and the increasing property of $\frac{G(t)}{t}$, it suffices to show that

$$\lim_{t\to\infty} \frac{G\left(t^{\frac{n}{n-1}}\right)}{t^{\frac{n}{n-1}}} \frac{t}{G(\lambda t)} = \infty \text{ for all } \lambda > 0, \tag{2.14}$$

for proving (2.13). Employing (2.12), we obtain for all fixed $\lambda > 0$ and sufficiently large t,

$$\frac{G(t^{\frac{n}{n-1}})}{G(\lambda t)} \geq \frac{t^{\frac{1+\alpha}{n-1}}}{\lambda^{1+\alpha}}.$$

Thus, we have

$$\lim_{t\to\infty} \frac{G(t^{\frac{n}{n-1}})}{G(\lambda t)} \frac{t}{t^{\frac{n}{n-1}}} \geq \lim_{t\to\infty} \frac{t^{\frac{\alpha}{n-1}}}{\lambda^{1+\alpha}} = \infty,$$

which implies (2.14). Thus, $G \preceq G_T$.

If G satisfies (2.8), then the trace operator, $T : W^{1,G}(\Omega) \to L^G(\partial\Omega)$, is compact by Lemma 2.2.4 and [2, Theorem 8.25]. On the other hand, if G satisfies (2.9), then the trace operator, $T : W^{1,G}(\Omega) \to L^G(\partial\Omega)$, is also compact by Lemma 2.2.4, [2, Theorem 8.25] and from the fact that L^∞ belongs to any Orlicz spaces. Therefore the trace operator, $T : W^{1,G}(\Omega) \to L^G(\partial\Omega)$, is compact for either case.

□

2.3 Proof of the main theorem

We remark that it suffices to prove this theorem when $x_0 = 0$, $r = 1$ and

$$\int_{\Omega\cap B_4} G(|Dv|)dx \leq 1,$$

as (2.1) is invariant under normalization, scaling and translation, see [12]. Changing this integration written as the polar coordinates and applying the

Chebyshev's inequality, we observe that for some $\rho \in (3,4)$

$$\int_{\Omega\cap\partial B_\rho} G(|Dv|)d\sigma \le c, \tag{2.15}$$

where σ is the Hausdorff measure of dimension $n-1$ and $c = c(G,n)$. Set $\phi_\tau(x) = \chi_\tau(|x|)$, $0 < \tau < 1$, where $\chi_\tau(t)$ is a continuous function such that $\chi_\tau(t) = 0$ for $t \in [0, \rho-\tau]$, linear for $t \in [\rho-\tau, \rho]$, and $\chi_\tau(t) = 1$ for $t \in [\rho, \infty)$. Taking this ϕ_τ as a test function in (2.1) and letting $\tau \to 0$, we see that

$$\int_{\Omega\cap\partial B_\rho} \frac{g(|Dv|)}{|Dv|} v_\nu d\sigma = 0, \tag{2.16}$$

where ν is the outward unit normal to $\Omega \cap \partial B_\rho$ and $v_\nu = D_\nu v$. For any fixed $0 < \epsilon < 1$, there exists a C^2-convex domain V_ϵ such that $\Omega \subset V_\epsilon$ and $dist(\partial\Omega, \partial V_\epsilon) < \frac{\epsilon}{10}$, as follows from the Lemma 3.2.1.1 in [41]. Set $\Omega_\epsilon = \left\{x \in \mathbb{R}^n : dist(x, V_\epsilon) < \frac{9}{10}\epsilon\right\}$. Now we define

$$f(x) = \begin{cases} 0 & \text{if } x \in \big((\Omega_\epsilon \setminus \Omega) \cap \partial B_\rho\big) \cup (\partial\Omega_\epsilon \cap B_\rho) \\ \frac{g(|Dv|)}{|Dv|} v_\nu & \text{if } x \in \Omega \cap \partial B_\rho, \end{cases}$$

and

$$\mathcal{F}^\epsilon(w) = \int_{\Omega_\epsilon\cap B_\rho} G\left(\sqrt{\epsilon + |Dw|^2}\right) dx - \int_{\Omega\cap\partial B_\rho} f w \, d\sigma$$

for $w \in W^{1,G}(\Omega_\epsilon \cap B_\rho)$, where w in the second integration means the trace of w. Let v^ϵ be a minimum of the functional $\mathcal{F}^\epsilon(w)$ in $\{w \in W^{1,G}(\Omega_\epsilon \cap B_\rho) : \int_{\Omega_\epsilon\cap B_\rho} w d\sigma = 0\}$. In fact, by weakly lower semicontinuity of $\mathcal{F}^\epsilon(w)$ and compactness of the trace operator from corollary 2.2.7, we conclude that such v^ϵ exists. Then v^ϵ satisfies

$$\int_{\Omega_\epsilon\cap B_\rho} \frac{g\left(\sqrt{\epsilon + |Dv^\epsilon|^2}\right)}{\sqrt{\epsilon + |Dv^\epsilon|^2}} Dv^\epsilon \cdot D\varphi \, dx = \int_{\Omega\cap\partial B_\rho} f\varphi d\sigma \tag{2.17}$$

for all $\varphi \in W^{1,G}(\Omega_\epsilon \cap B_\rho)$. Taking $\varphi = v^\epsilon$ in (2.17), we estimate the right-hand side employing Young's inequality and Poincaré inequality as follows:

$$\begin{aligned}\int_{\Omega\cap\partial B_\rho} fv^\epsilon d\sigma &\leq \int_{\Omega\cap\partial B_\rho} g(|Dv|)|v^\epsilon| d\sigma \\ &\leq \int_{\Omega\cap\partial B_\rho} G^*(g(|Dv|)) + G(|v^\epsilon|) d\sigma \\ &= \int_{\Omega\cap\partial B_\rho} ((|Dv|g(|Dv|) - G(|Dv|)) + G(|v^\epsilon|) d\sigma \\ &\leq \int_{\Omega\cap\partial B_\rho} s_g G(|Dv|) d\sigma + \int_{\Omega\cap\partial B_\rho} G(|v^\epsilon|) d\sigma.\end{aligned}$$

To estimate $\int_{\Omega\cap\partial B_\rho} G(|v^\epsilon|)d\sigma$, let's assume $k =: \|v^\epsilon\|_{L^G(\Omega_\epsilon\cap\partial B_\rho)} > 1$. Then we have

$$G\left(\sqrt{\epsilon + |Dv^\epsilon|^2}\right) \geq G\left(\sqrt{\epsilon + \left(\frac{|Dv^\epsilon|}{k}\right)^2}\right),$$

which deduces that $\mathcal{F}^\epsilon\left(\frac{v^\epsilon}{k}\right) \leq \mathcal{F}^\epsilon(v^\epsilon)$ if $\int_{\Omega_\epsilon\cap\partial B_\rho} fv^\epsilon\, d\sigma < 0$, and $\mathcal{F}^\epsilon\left(-\frac{v^\epsilon}{k}\right) \leq \mathcal{F}^\epsilon(v^\epsilon)$ else if $\int_{\Omega_\epsilon\cap\partial B_\rho} fv^\epsilon\, d\sigma \geq 0$. This contradiction establishes that $\|v^\epsilon\|_{L^G(\Omega_\epsilon\cap\partial B_\rho)} \leq 1$, and then $\int_{\Omega\cap\partial B_\rho} G(|v^\epsilon|)d\sigma \leq 1$, see [65, Theorem 3.2.2]. Taking into account the above estimates and (2.15), we obtain

$$\int_{\Omega_\epsilon\cap B_\rho} \frac{g\left(\sqrt{\epsilon + |Dv^\epsilon|^2}\right)}{\sqrt{\epsilon + |Dv^\epsilon|^2}} |Dv^\epsilon|^2\, dx \leq c, \tag{2.18}$$

where $c = c(G, n)$. We then find

$$\begin{aligned}\int_{\Omega_\epsilon\cap B_\rho} G(\sqrt{\epsilon+|Dv^\epsilon|^2})dx &\le c\int_{\Omega_\epsilon\cap B_\rho\cap\{\epsilon\le|Dv^\epsilon|^2\}} \frac{g\left(\sqrt{\epsilon+|Dv^\epsilon|^2}\right)}{\sqrt{\epsilon+|Dv^\epsilon|^2}}\left(\epsilon+|Dv^\epsilon|^2\right)dx\\ &+c\int_{\Omega_\epsilon\cap B_\rho\cap\{\epsilon>|Dv^\epsilon|^2\}} g\left(\sqrt{\epsilon+|Dv^\epsilon|^2}\right)\sqrt{\epsilon+|Dv^\epsilon|^2}dx\\ &\le c\int_{\Omega_\epsilon\cap B_\rho\cap\{\epsilon\le|Dv^\epsilon|^2\}} \frac{g\left(\sqrt{\epsilon+|Dv^\epsilon|^2}\right)}{\sqrt{\epsilon+|Dv^\epsilon|^2}}|Dv^\epsilon|^2dx\\ &+c\int_{\Omega_\epsilon\cap B_\rho\cap\{\epsilon>|Dv^\epsilon|^2\}} g(\sqrt{2\epsilon})\sqrt{2\epsilon}\,dx\\ &\le c,\end{aligned} \tag{2.19}$$

where we have used (2.4), (2.18) and the assumptions that g is increasing and $0<\epsilon<1$. According to the main result of [28], $v^\epsilon \in C^{1,\alpha}_{loc}(\Omega_\epsilon\cap B_\rho)$ for some exponent $0<\alpha<1$. We then deduce from the main result of [25] that

$$v^\epsilon \in W^{2,2}_{loc}(\Omega_\epsilon \cap B_\rho).$$

Thus one can deduce from (2.17) that v^ϵ solves

$$\operatorname{div}\left(\frac{g(\sqrt{\epsilon+|Dv^\epsilon|^2})}{\sqrt{\epsilon+|Dv^\epsilon|^2}}Dv^\epsilon\right)=0 \text{ a.e. in } \Omega_{\tilde\epsilon}\cap B_2, \tag{2.20}$$

where $\tilde\epsilon\in(0,\epsilon)$ is to be selected later. Let $\eta\in\mathbb{R}^n$ with $|\eta|=1$, and let $v_\eta = D_\eta v$ denote the directional derivative of v in the direction of η. Taking the directional derivative of (2.20) in the direction of η, we see that ${v^\epsilon}_\eta$ is a weak solution to

$$\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(b_{ij}\frac{\partial {v^\epsilon}_\eta}{\partial x_j}\right)=0 \quad \text{in } \Omega_{\tilde\epsilon}\cap B_2, \tag{2.21}$$

where

$$b_{ij}=\left(\frac{g'(h^\epsilon)}{(h^\epsilon)^2}-\frac{g(h^\epsilon)}{(h^\epsilon)^3}\right){v^\epsilon}_{x_i}{v^\epsilon}_{x_j}+\frac{g(h^\epsilon)}{h^\epsilon}\delta_{ij} \text{ and } h^\epsilon=\sqrt{\epsilon+|Dv^\epsilon|^2}.$$

From [25, Lemma 2.2], we know that

$$min(1, i_g)\frac{g(h^\epsilon)}{h^\epsilon}|\xi|^2 \leq b_{ij}\xi_i\xi_j \leq max(1, s_g)\frac{g(h^\epsilon)}{h^\epsilon}|\xi|^2 \tag{2.22}$$

for all $\xi \in \mathbb{R}^n$. Let

$$c_{ij} = \frac{h^\epsilon}{g(h^\epsilon)}b_{ij} = \left(\frac{g'(h^\epsilon)}{h^\epsilon g(h^\epsilon)} - \frac{1}{(h^\epsilon)^2}\right){v^\epsilon}_{x_i}{v^\epsilon}_{x_j} + \delta_{ij}.$$

By (2.22), we see that

$$min(1, i_g)|\xi|^2 \leq c_{ij}\xi_i\xi_j \leq max(1, s_g)|\xi|^2 \tag{2.23}$$

for all $\xi \in \mathbb{R}^n$. We write

$$L_\phi(G(h^\epsilon)) = \sum_{i,j=1}^{n}\int_{\partial(\Omega_{\tilde{\epsilon}}\cap B_2)} c_{ij}G(h^\epsilon)_{x_j}\phi\nu_i d\sigma - \sum_{i,j=1}^{n}\int_{\Omega_{\tilde{\epsilon}}\cap B_2} c_{ij}G(h^\epsilon)_{x_j}\phi_{x_i}dx \tag{2.24}$$

for some fixed $\phi \in C^\infty(\overline{\Omega_{\tilde{\epsilon}} \cap B_2}) \cap C^\infty(B_2)$ with $\phi \geq 0$.

We now claim that there is $\tilde{\epsilon} \in (0, \epsilon)$ for which the first integration in the right-hand side is well-defined. Observe that

$$\begin{aligned}\int_{\Omega_{\frac{1}{2}\epsilon}\cap B_2} G(h^\epsilon)_{x_j}dx &= \sum_{k=1}^{n}\int_{\Omega_{\frac{1}{2}\epsilon}\cap B_2}\frac{g(h^\epsilon)}{h^\epsilon}{v^\epsilon}_{x_k}{v^\epsilon}_{x_kx_j} \\ &\leq \int_{\Omega_{\frac{1}{2}\epsilon}\cap B_2}\frac{g(h^\epsilon)}{h^\epsilon}|Dv^\epsilon|^2dx + \int_{\Omega_{\frac{1}{2}\epsilon}\cap B_2}\frac{g(h^\epsilon)}{h^\epsilon}|D^2v^\epsilon|^2dx,\end{aligned}$$

which is finite by (2.18) and [25, Theorem 2.3]. Hence $\int_{\partial\Omega_{\tilde{\epsilon}}\cap B_2} G(h^\epsilon)_{x_j}d\sigma$ is finite for a.e. $\tilde{\epsilon} \in \left(0, \frac{1}{2}\epsilon\right)$ by Fubini's theorem. It implies that the first integration in the right-hand side of (2.24) can be well defined for some $\tilde{\epsilon} \in \left(0, \frac{1}{2}\epsilon\right)$ and we choose that $\tilde{\epsilon}$.

The second integration in the right-hand side of (2.24) is computed as follows,

$$
\begin{aligned}
&\sum_{i,j=1}^{n}\int_{\Omega_{\tilde{\epsilon}}\cap B_2} c_{ij}G(h^\epsilon)_{x_j}\phi_{x_i}dx\\
&\qquad=\sum_{i,j=1}^{n}\int_{\Omega_{\tilde{\epsilon}}\cap B_2}\left(\frac{g'(h^\epsilon)(h^\epsilon)^2{h^\epsilon}_{x_j}-g(h^\epsilon)h^\epsilon {h^\epsilon}_{x_j}}{(h^\epsilon)^3}{v^\epsilon}_{x_i}{v^\epsilon}_{x_j}\right)\phi_{x_i}dx\\
&\qquad\quad+\sum_{i=1}^{n}\int_{\Omega_{\tilde{\epsilon}}\cap B_2} G(h^\epsilon)_{x_i}\phi_{x_i}dx \qquad (2.25)\\
&\qquad=\underbrace{\sum_{i,j,k=1}^{n}\int_{\Omega_{\tilde{\epsilon}}\cap B_2}\left(\frac{g'(h^\epsilon)h^\epsilon-g(h^\epsilon)}{(h^\epsilon)^3}{v^\epsilon}_{x_k}{v^\epsilon}_{x_kx_j}{v^\epsilon}_{x_i}{v^\epsilon}_{x_j}\right)\phi_{x_i}dx}_{I}\\
&\qquad\quad+\sum_{i=1}^{n}\int_{\Omega_{\tilde{\epsilon}}\cap B_2} G(h^\epsilon)_{x_i}\phi_{x_i}dx.
\end{aligned}
$$

An integration by parts reveals

$$
\begin{aligned}
I=&\sum_{i,j,k=1}^{n}\int_{\partial(\Omega_{\tilde{\epsilon}}\cap B_2)}\frac{g'(h^\epsilon)h^\epsilon-g(h^\epsilon)}{(h^\epsilon)^3}{v^\epsilon}_{x_k}{v^\epsilon}_{x_kx_j}{v^\epsilon}_{x_j}{v^\epsilon}_{x_i}\phi\nu_i d\sigma\\
&-\sum_{i,j,k=1}^{n}\int_{\Omega_{\tilde{\epsilon}}\cap B_2}\frac{g'(h^\epsilon)h^\epsilon-g(h^\epsilon)}{(h^\epsilon)^3}{v^\epsilon}_{x_jx_i}{v^\epsilon}_{x_i}{v^\epsilon}_{x_k}{v^\epsilon}_{x_kx_j}\phi\,dx \qquad (2.26)\\
&\underbrace{-\sum_{i,j,k=1}^{n}\int_{\Omega_{\tilde{\epsilon}}\cap B_2}{v^\epsilon}_{x_j}\left(\frac{g'(h^\epsilon)h^\epsilon-g(h^\epsilon)}{(h^\epsilon)^3}{v^\epsilon}_{x_i}{v^\epsilon}_{x_k}{v^\epsilon}_{x_kx_j}\right)_{x_i}\phi\,dx}_{II}.
\end{aligned}
$$

Choosing a test function ${v^\epsilon}_{x_j}\phi$, replacing j with k and substituting e_j for η in (2.21), we obtain

$$
\sum_{i,j,k=1}^{n}\int_{\Omega_{\tilde{\epsilon}}\cap B_2}\left(\frac{g'(h^\epsilon)h^\epsilon-g(h^\epsilon)}{(h^\epsilon)^3}{v^\epsilon}_{x_i}{v^\epsilon}_{x_k}{v^\epsilon}_{x_kx_j}+\frac{g(h^\epsilon)}{h^\epsilon}{v^\epsilon}_{x_kx_j}\delta_{ik}\right)_{x_i}{v^\epsilon}_{x_j}\phi\,dx=0.
$$

This implies

$$\sum_{i,j,k=1}^{n}\int_{\Omega_{\tilde\epsilon}\cap B_2}\left(\frac{g'(h^\epsilon)h^\epsilon-g(h^\epsilon)}{(h^\epsilon)^3}{v^\epsilon}_{x_i}{v^\epsilon}_{x_k}{v^\epsilon}_{x_kx_j}\right)_{x_i}{v^\epsilon}_{x_j}\phi\,dx$$
$$=-\sum_{i,j=1}^{n}\int_{\Omega_{\tilde\epsilon}\cap B_2}\left(\frac{g(h^\epsilon)}{h^\epsilon}{v^\epsilon}_{x_ix_j}\right)_{x_i}{v^\epsilon}_{x_j}\phi\,dx.$$

Applying this identity to II and using an integration by parts, we have

$$\begin{aligned}
II&=-\sum_{i,j=1}^{n}\int_{\Omega_{\tilde\epsilon}\cap B_2}\left(\frac{g(h^\epsilon)}{h^\epsilon}{v^\epsilon}_{x_ix_j}\right)_{x_i}{v^\epsilon}_{x_j}\phi\,dx\\
&=-\sum_{i,j=1}^{n}\int_{\partial(\Omega_{\tilde\epsilon}\cap B_2)}\frac{g(h^\epsilon)}{h^\epsilon}{v^\epsilon}_{x_ix_j}{v^\epsilon}_{x_j}\phi\nu_i\,d\sigma\\
&\quad+\sum_{i,j=1}^{n}\int_{\Omega_{\tilde\epsilon}\cap B_2}{v^\epsilon}_{x_jx_i}\frac{g(h^\epsilon)}{h^\epsilon}{v^\epsilon}_{x_jx_i}\phi\,dx+\sum_{i,j=1}^{n}\int_{\Omega_{\tilde\epsilon}\cap B_2}\frac{g(h^\epsilon)}{h^\epsilon}{v^\epsilon}_{x_j}{v^\epsilon}_{x_jx_i}\phi_{x_i}dx\\
&=-\sum_{i,j=1}^{n}\int_{\partial(\Omega_{\tilde\epsilon}\cap B_2)}\frac{g(h^\epsilon)}{h^\epsilon}{v^\epsilon}_{x_ix_j}{v^\epsilon}_{x_j}\phi\nu_i\,d\sigma\\
&\quad+\sum_{i,j=1}^{n}\int_{\Omega_{\tilde\epsilon}\cap B_2}{v^\epsilon}_{x_jx_i}\frac{g(h^\epsilon)}{h^\epsilon}{v^\epsilon}_{x_jx_i}\phi dx+\sum_{i=1}^{n}\int_{\Omega_{\tilde\epsilon}\cap B_2}G(h^\epsilon)_{x_i}\phi_{x_i}dx.
\end{aligned}\tag{2.27}$$

Combining (2.24), (2.25), (2.26) and (2.27), we find

$$\begin{aligned}
&L_\phi\big(G(h^\epsilon)\big)-\sum_{i,j=1}^{n}\int_{\partial(\Omega_{\tilde\epsilon}\cap B_2)}c_{ij}G(h^\epsilon)_{x_j}\phi\nu_id\sigma\\
&=-\sum_{i,j,k=1}^{n}\int_{\partial(\Omega_{\tilde\epsilon}\cap B_2)}\frac{g'(h^\epsilon)h^\epsilon-g(h^\epsilon)}{(h^\epsilon)^3}{v^\epsilon}_{x_k}{v^\epsilon}_{x_kx_j}{v^\epsilon}_{x_j}{v^\epsilon}_{x_i}\phi\nu_id\sigma\\
&\quad+\sum_{i,j,k=1}^{n}\int_{\Omega_{\tilde\epsilon}\cap B_2}\frac{g'(h^\epsilon)h^\epsilon-g(h^\epsilon)}{(h^\epsilon)^3}{v^\epsilon}_{x_jx_i}{v^\epsilon}_{x_i}{v^\epsilon}_{x_k}{v^\epsilon}_{x_kx_j}\phi\,dx\\
&\quad-\sum_{i,j=1}^{n}\int_{\partial(\Omega_{\tilde\epsilon}\cap B_2)}\frac{g(h^\epsilon)}{h^\epsilon}{v^\epsilon}_{x_ix_j}{v^\epsilon}_{x_j}\phi\nu_i\,d\sigma+\sum_{i,j=1}^{n}\int_{\Omega_{\tilde\epsilon}\cap B_2}\frac{g(h^\epsilon)}{h^\epsilon}({v^\epsilon}_{x_jx_i})^2\phi\,dx.
\end{aligned}$$

Employing (2.22) and the fact that

$$|h^\epsilon{}_{x_k}| = \frac{|Dv^\epsilon \cdot Dv^\epsilon{}_{x_k}|}{\sqrt{\epsilon + |Dv^\epsilon|^2}} \leq \frac{|Dv^\epsilon||Dv^\epsilon{}_{x_k}|}{\sqrt{\epsilon + |Dv^\epsilon|^2}} \leq |Dv^\epsilon{}_{x_k}|,$$

we discover

$$\begin{aligned} L_\phi(G(h^\epsilon)) &- \sum_{i,j=1}^n \int_{\partial(\Omega_{\tilde{\epsilon}} \cap B_2)} c_{ij} G(h^\epsilon)_{x_j} \phi \nu_i d\sigma \\ &+ \sum_{i,j,k=1}^n \int_{\partial(\Omega_{\tilde{\epsilon}} \cap B_2)} \frac{g'(h^\epsilon)h^\epsilon - g(h^\epsilon)}{(h^\epsilon)^3} v^\epsilon{}_{x_k} v^\epsilon{}_{x_k x_j} v^\epsilon{}_{x_j} v^\epsilon{}_{x_i} \phi \nu_i d\sigma \\ &+ \sum_{i,j=1}^n \int_{\partial(\Omega_{\tilde{\epsilon}} \cap B_2)} \frac{g(h^\epsilon)}{h^\epsilon} v^\epsilon{}_{x_i x_j} v^\epsilon{}_{x_j} \phi \nu_i \, d\sigma \\ &\geq \min(1, i_g) \sum_{k=1}^n \int_{\Omega_{\tilde{\epsilon}} \cap B_2} \frac{g(h^\epsilon)}{h^\epsilon} |Dv^\epsilon{}_{x_k}|^2 \phi \, dx \\ &\geq \min(1, i_g) \int_{\Omega_{\tilde{\epsilon}} \cap B_2} \frac{g(h^\epsilon)}{h^\epsilon} |Dh^\epsilon|^2 \phi \, dx. \end{aligned} \tag{2.28}$$

We recall the following inequalities that can be derived from [56, Lemma 1.1],

$$\frac{1}{t} \leq \frac{g(t)}{G(t)} \leq \frac{1 + s_g}{t} \text{ for } t > 0.$$

This inequality leads us to the following estimate,

$$\left| D\left(\sqrt{G(h^\epsilon)}\right)\right|^2 = \left|\frac{g(h^\epsilon)}{2\sqrt{G(h^\epsilon)}} Dh^\epsilon\right|^2 = \frac{g^2(h^\epsilon)}{4G(h^\epsilon)}|Dh^\epsilon|^2 \leq \frac{(1+s_g)g(h^\epsilon)}{4h^\epsilon}|Dh^\epsilon|^2.$$

Applying this to (2.28), we have

$$\begin{aligned}
&\frac{4\min(1,i_g)}{1+s_g}\int_{\Omega_{\tilde{\epsilon}}\cap B_2}\left|D\left(\sqrt{G(h^\epsilon)}\right)\right|^2\phi\,dx \qquad (2.29)\\
&\quad\le L_\phi\big(G(h^\epsilon)\big)+\sum_{i,j,k=1}^{n}\int_{\partial(\Omega_{\tilde{\epsilon}}\cap B_2)}\frac{g'(h^\epsilon)h^\epsilon-g(h^\epsilon)}{(h^\epsilon)^3}{v^\epsilon}_{x_k}{v^\epsilon}_{x_kx_j}{v^\epsilon}_{x_j}{v^\epsilon}_{x_i}\phi\nu_i d\sigma.\\
&\quad-\sum_{i,j=1}^{n}\int_{\partial(\Omega_{\tilde{\epsilon}}\cap B_2)}c_{ij}G(h^\epsilon)_{x_j}\phi\nu_i d\sigma+\sum_{i,j=1}^{n}\int_{\partial(\Omega_{\tilde{\epsilon}}\cap B_2)}\frac{g(h^\epsilon)}{h^\epsilon}{v^\epsilon}_{x_ix_j}{v^\epsilon}_{x_j}\phi\nu_i\,d\sigma.
\end{aligned}$$

Let $\phi=\varphi^2$ for $\varphi\in C_0^\infty(B_2)$. Employing (2.29) and (2.24), we see that

$$\begin{aligned}
&\frac{4\min(1,i_g)}{1+s_g}\int_{\Omega_{\tilde{\epsilon}}\cap B_2}\left|D\left(\sqrt{G(h^\epsilon)}\right)\right|^2\varphi^2\,dx\\
&\le L_{\varphi^2}\left(G(h^\epsilon)\right)+\sum_{i,j,k=1}^{n}\int_{\partial(\Omega_{\tilde{\epsilon}}\cap B_2)}\frac{g'(h^\epsilon)h^\epsilon-g(h^\epsilon)}{(h^\epsilon)^3}{v^\epsilon}_{x_k}{v^\epsilon}_{x_kx_j}{v^\epsilon}_{x_j}\varphi^2{v^\epsilon}_{x_i}\nu_i d\sigma\\
&-\sum_{i,j=1}^{n}\int_{\partial(\Omega_{\tilde{\epsilon}}\cap B_2)}c_{ij}G(h^\epsilon)_{x_j}\varphi^2\nu_i d\sigma+\sum_{i,j=1}^{n}\int_{\partial(\Omega_{\tilde{\epsilon}}\cap B_2)}\frac{g(h^\epsilon)}{h^\epsilon}{v^\epsilon}_{x_ix_j}{v^\epsilon}_{x_j}\varphi^2\nu_i\,d\sigma\\
&=\sum_{i,j,k=1}^{n}\int_{\partial\Omega_{\tilde{\epsilon}}\cap B_2}\frac{g'(h^\epsilon)h^\epsilon-g(h^\epsilon)}{(h^\epsilon)^3}{v^\epsilon}_{x_k}{v^\epsilon}_{x_kx_j}{v^\epsilon}_{x_j}{v^\epsilon}_{x_i}\varphi^2\nu_i d\sigma \qquad (2.30)\\
&+\sum_{i,j=1}^{n}\int_{\partial\Omega_{\tilde{\epsilon}}\cap B_2}\frac{g(h^\epsilon)}{h^\epsilon}{v^\epsilon}_{x_ix_j}{v^\epsilon}_{x_j}\varphi^2\nu_i\,d\sigma-2\sum_{i,j=1}^{n}\int_{\Omega_{\tilde{\epsilon}}\cap B_2}\varphi\varphi_{x_i}c_{ij}(G(h^\epsilon))_{x_j}dx\\
&:=\mathrm{III}+\mathrm{IV}+\mathrm{V}.
\end{aligned}$$

From (2.20) and (2.17), it can be derived that

$$\begin{aligned}
0&=\int_{\Omega_{\tilde{\epsilon}}\cap B_2}\operatorname{div}\left(\frac{g(h^\epsilon)}{h^\epsilon}Dv^\epsilon\right)\psi dx\\
&=\int_{\partial\Omega_{\tilde{\epsilon}}\cap B_2}\psi\frac{g(h^\epsilon)}{h^\epsilon}Dv^\epsilon\cdot\nu d\sigma-\int_{\Omega_{\tilde{\epsilon}}\cap B_2}\frac{g(h^\epsilon)}{h^\epsilon}Dv^\epsilon\cdot D\psi dx\\
&=\int_{\partial\Omega_{\tilde{\epsilon}}\cap B_2}\psi\frac{g(h^\epsilon)}{h^\epsilon}Dv^\epsilon\cdot\nu d\sigma
\end{aligned}$$

for any $\psi\in C_0^\infty(B_2)$. Then, since $\frac{g(h^\epsilon)}{h^\epsilon}>0$, we find that $D_\nu v^\epsilon=0$ a.e. on

$\partial\Omega_{\tilde{\epsilon}} \cap B_2$, and so III $= 0$. Employing [41, Eq. (3.1.1.8)], we have

$$\begin{aligned}
\mathrm{IV} &= \sum_{i,j=1}^{n} \int_{\partial\Omega_{\tilde{\epsilon}}\cap B_2} \frac{g(h^\epsilon)}{h^\epsilon} {v^\epsilon}_{x_i x_j} {v^\epsilon}_{x_j} \varphi^2 \nu_i \, d\sigma \\
&= \int_{\partial\Omega_{\tilde{\epsilon}}\cap B_2} \varphi^2 \frac{g(h^\epsilon)}{h^\epsilon} \mathcal{B}(D_t v^\epsilon, D_t v^\epsilon) d\sigma,
\end{aligned}$$

where $\mathcal{B}(\cdot,\cdot)$ is the second fundamental quadratic form of $\partial\Omega_{\tilde{\epsilon}}$ and $D_t v^\epsilon$ is the tangential component of Dv^ϵ on $\partial\Omega_{\tilde{\epsilon}}$. Since $\Omega_{\tilde{\epsilon}}$ is convex, $\mathcal{B}(\cdot,\cdot) \le 0$. Hence, IV ≤ 0.

For V, using the boundedness of c_{ij}, the identity $\frac{g(h^\epsilon)}{2\sqrt{G(h^\epsilon)}} Dh^\epsilon = D\sqrt{G(h^\epsilon)}$, and Cauchy inequality with τ, we discover

$$\begin{aligned}
|\mathrm{V}| &\le 2 \sum_{i,j=1}^{n} \int_{\Omega_{\tilde{\epsilon}}\cap B_2} \left| \varphi \varphi_{x_i} c_{ij} \frac{g(h^\epsilon)}{2\sqrt{G(h^\epsilon)}} Dh^\epsilon \sqrt{G(h^\epsilon)} \right| dx \\
&\le \tau \int_{\Omega_{\tilde{\epsilon}}\cap B_2} \varphi^2 |D\sqrt{G(h^\epsilon)}|^2 dx + \frac{c}{\tau} \int_{\Omega_{\tilde{\epsilon}}\cap B_2} |D\varphi|^2 |G(h^\epsilon)| dx.
\end{aligned}$$

Taking τ small enough, we obtain from (2.30) that

$$\int_{\Omega_{\tilde{\epsilon}}\cap B_2} \varphi^2 |D\sqrt{G(h^\epsilon)}|^2 dx \le c \int_{\Omega_{\tilde{\epsilon}}\cap B_2} |D\varphi|^2 |G(h^\epsilon)| dx.$$

Then by applying Sobolev's inequality to $\varphi\sqrt{G(h^\epsilon)}$, we see that for $n \ne 2$,

$$\begin{aligned}
\|\varphi\sqrt{G(h^\epsilon)}\|^2_{L^{\frac{2n}{n-2}}(\Omega_{\tilde{\epsilon}}\cap B_2)} &\le c \|D(\varphi\sqrt{G(h^\epsilon)})\|^2_{L^2(\Omega_{\tilde{\epsilon}}\cap B_2)} \\
&\le c \int_{\Omega_{\tilde{\epsilon}}\cap B_2} |D\varphi|^2 G(h^\epsilon) dx.
\end{aligned} \tag{2.31}$$

In the case of $n = 2$, (2.31) is valid through replacing $\frac{2n}{n-2}$ by 4.

We next employ Moser iteration in a usual way to obtain that for a.e. $x \in \Omega_{\tilde{\epsilon}} \cap B_1$

$$G(\sqrt{\epsilon + |Dv^\epsilon|^2}) = G(h^\epsilon) \le c \int_{\Omega_{\tilde{\epsilon}}\cap B_2} G(h^\epsilon) dx = c \int_{\Omega_{\tilde{\epsilon}}\cap B_2} G(\sqrt{\epsilon + |Dv^\epsilon|^2}) dx, \tag{2.32}$$

where $c = c(n, i_g, s_g, |\Omega \cap B_2|)$. From (2.19), we know that the sequence $\{v^\epsilon\}$ is uniformly bounded in $W^{1,G}(\Omega \cap B_\rho)$. Thus there is a subsequence, still denoted by $\{v^\epsilon\}$, and a function $\tilde{v} \in W^{1,G}(\Omega \cap B_\rho)$ such that

$$v^\epsilon \to \tilde{v} \in L^G(\Omega \cap B_\rho) \text{ and } Dv^\epsilon \rightharpoonup D\tilde{v} \in L^G(\Omega \cap B_\rho) \text{ as } \epsilon \to 0.$$

On the other hand, it follows from (2.32) and (2.19) that $G\left(\sqrt{\epsilon + |Dv^\epsilon|^2}\right)$ is uniformly bounded on compact subsets of $\Omega_{\tilde{\epsilon}} \cap B_\rho$. In addition, we recall that Dv^ϵ is Hölder continuous on compact subsets of $\Omega_{\tilde{\epsilon}} \cap B_\rho$ with Hölder exponent independent of ϵ (see [28]). Then we apply Arzelá-Ascoli theorem to discover that $\{v^\epsilon\}$ and $\{Dv^\epsilon\}$ converge uniformly up to a subsequence, on compact subsets of $\Omega \cap B_\rho$ as $\epsilon \to 0$. Putting

$$\tilde{g}_\epsilon = \begin{cases} \frac{g\left(\sqrt{\epsilon+|Dv^\epsilon|^2}\right)}{\sqrt{\epsilon+|Dv^\epsilon|^2}} Dv^\epsilon \cdot D\tilde{\varphi} & \text{in } \Omega_\epsilon \cap B_\rho \\ 0 & \text{in } (\Omega_\epsilon \cap B_\rho)^c \end{cases}$$

for some $\tilde{\varphi} \in C^\infty(\mathbb{R}^n)$, it follows from Young's inequality, (2.6), (2.4) and (2.19) that

$$\begin{aligned} \int_{\Omega_1 \cap B_\rho} G^*\left(\tilde{g}_\epsilon\right) dx &= \int_{\Omega_\epsilon \cap B_\rho} G^*\left(\frac{g\left(\sqrt{\epsilon + |Dv^\epsilon|^2}\right)}{\sqrt{\epsilon + |Dv^\epsilon|^2}} Dv^\epsilon \cdot D\tilde{\varphi}\right) dx \\ &\le M^{k_1^*} \int_{\Omega_\epsilon \cap B_\rho} G^*\left(g\left(\sqrt{\epsilon + |Dv^\epsilon|^2}\right)\right) dx \\ &\le M^{k_1^*} s_g \int_{\Omega_\epsilon \cap B_\rho} G\left(\sqrt{\epsilon + |Dv^\epsilon|^2}\right) dx \\ &\le c, \end{aligned}$$

where $M = max(\|D\tilde{\varphi}\|_{L^\infty(\Omega_\epsilon \cap B_\rho)}, 1)$ and $k_1^* = \Delta_2(G^*)$. Consequently $\{\tilde{g}_\epsilon\}_\epsilon$ is uniformly integrable by de la Vallée Poussin theorem and then, employing Vitali convergence theorem to (2.17), we see that following relation holds

$$\int_{\Omega \cap B_\rho} \frac{g(|D\tilde{v}|)}{|D\tilde{v}|} D\tilde{v} \cdot D\tilde{\varphi}\, dx = \int_{\Omega \cap \partial B_\rho} f\tilde{\varphi} d\sigma \qquad (2.33)$$

for $\tilde{\varphi} \in C^\infty(\mathbb{R}^n)$. Since $\Omega \cap B_\rho$ is a bounded Lipschitz domain, it has the restricted cone property by [37, Theorem 4.3]. Then we can find an extension operator for $W^{1,G}(\Omega \cap B_\rho)$, see [35, Theorem 3.3.2], which means that any function in $W^{1,G}(\Omega \cap B_\rho)$ can be approximated by functions in $C^\infty(\mathbb{R}^n)$. Hence (2.33) holds for $\tilde{\varphi} \in W^{1,G}(\Omega \cap B_\rho)$. Since v is the unique weak solution up to a constant, $v - \tilde{v}$ is a constant. Accordingly, (2.32) is true for $\epsilon = 0$ when v^ϵ and $\Omega_{\tilde{\epsilon}}$ are replaced by v and Ω, respectively. Now the main estimate (2.5) follows.

Chapter 3

Gradient estimates for elliptic equations in quasiconvex domains

3.1 Overview

We study nonlinear Calderón-Zygmund type estimates for the weak solution to the following Dirichlet problem:

$$\begin{cases} \operatorname{div} \mathbf{a}(x, Du) = \operatorname{div}\left(|\mathbf{f}|^{p-2}\mathbf{f}\right) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.1}$$

for $1 < p < \infty$. Here Ω is a bounded domain in $\mathbb{R}^n$, $\mathbf{f}$ is a given vector-valued function in $L^p(\Omega; \mathbb{R}^n)$, and the coefficient $\mathbf{a} = \mathbf{a}(x, \xi) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is another vector-valued function that is measurable in x and differentiable in ξ. As basic structure conditions, we impose the following assumptions on $\mathbf{a}$, namely ellipticity and growth conditions:

$$\langle D_\xi \mathbf{a}(x, \xi)\eta, \, \eta \rangle \geq \lambda |\xi|^{p-2} |\eta|^2 \quad \text{and} \tag{3.2}$$

$$|\mathbf{a}(x, \xi)| + |\xi||D_\xi \mathbf{a}(x, \xi)| \leq \Lambda |\xi|^{p-1}, \tag{3.3}$$

for each $\xi, \eta \in \mathbb{R}^n$, a.e. $x \in \mathbb{R}^n$, and some positive constants λ and Λ with $\lambda \leq 1 \leq \Lambda$.

It is easy to check that the assumptions (3.2) and (3.3) imply the following monotonicity condition:

$$\langle \mathbf{a}(x,\xi)-\mathbf{a}(x,\eta),\, \xi-\eta\rangle \geq \begin{cases} \tilde{\lambda}|\xi-\eta|^p & \text{if} \quad p\geq 2 \\ \tilde{\lambda}|\xi-\eta|^2(|\xi|+|\eta|)^{p-2} & \text{if} \quad 1<p<2, \end{cases} \tag{3.4}$$

where $\tilde{\lambda}$ is a positive constant depending only on λ and p.

Now we define a weak solution to the problem (3.1), as usual.

Definition 3.1.1. $u \in W_0^{1,p}(\Omega)$ is said to be a weak solution to (3.1), if it satisfies

$$\int_\Omega \langle \mathbf{a}(x,Du),\, D\phi\rangle\, dx = \int_\Omega \langle |\mathbf{f}|^{p-2}\mathbf{f},\, D\phi\rangle\, dx \quad \text{for all} \quad \phi\in W_0^{1,p}(\Omega). \tag{3.5}$$

By the method of Browder and Minty (see [54]), it is well known from that under the basic assumptions (3.2) and (3.3), the problem (3.1) has a unique weak solution provided $\mathbf{f}\in L^p(\Omega,\mathbb{R}^n)$ and $|\Omega|<\infty$, with the estimate

$$\|Du\|_{L^p(\Omega,\mathbb{R}^n)} \leq c\|\mathbf{f}\|_{L^p(\Omega,\mathbb{R}^n)}, \tag{3.6}$$

for some positive constant $c=c(\lambda,p)$. The main purpose of this paper is to establish a global Calderón-Zygmund theory, in short

$$\mathbf{f}\in L^q(\Omega,\mathbb{R}^n) \Rightarrow Du\in L^q(\Omega,\mathbb{R}^n) \text{ for all } q\in[p,\infty). \tag{3.7}$$

In particular, we are interested in the Calderón-Zygmund estimate like

$$\|Du\|_{L^q(\Omega,\mathbb{R}^n)} \leq c\|\mathbf{f}\|_{L^q(\Omega,\mathbb{R}^n)} \text{ for all } q\in[p,\infty), \tag{3.8}$$

the constant c being independent of u and $\mathbf{f}$.

In this direction, there are two important issues. One is the smoothness of the coefficient and the other is the geometry of $\partial\Omega$, as the Calderón-Zygmund theory usually requires more regularity on $\mathbf{a}$ and a suitable geometric condition on $\partial\Omega$. In our work, we also focus on those two issues, as we try to find optimal conditions on $\mathbf{a}$ and $\partial\Omega$ under which the weak solution satisfies (3.7) with the estimate(3.8).

In this respect, there have been plenty of results including [11, 13, 17, 18, 20, 63, 31, 61, 64]. For instance, Byun and Wang in [17] studied the special case ($p=2$) of (3.1) to establish the global Calderón-Zygmund estimate. They proved (3.7) under the conditions that $\mathbf{a}$ satisfies a certain vanishing condition described in Definition 3.1.2 and $\partial\Omega$ is (δ, R)-Reifenberg flat. As for the domain issue, the concept of Reifenberg flatness is already so general that it includes very rough domains like Koch snowflake. We refer to [13, 27] for the precise concept of Reifenberg flat domains. However, Reifenberg flatness excludes some geometrical simple domains, such as polygons. Here we deal with the so-called quasiconvex domains which include the polygons. A quasiconvex domain is a natural extension of a convex one(see Definition 3.1.3). On the other hand, there have been interesting regularity results on convex domains in the different context. Adolfsson in [6] studied the global Calderón-Zygmund estimate for green potential with $1 < p < 2$ on the convex domains. In [22, 23, 24] Cianchi and Maz'ya considered quasilinear elliptic equations and systems on the convex domains for Lipschitz regularity.

For the regularity assumption on the coefficients, we keep using the vanishing condition previously used in [17, 19].

Definition 3.1.2. (Small BMO-Seminorm Assumption)
We say that $\mathbf{a}$ is (δ, R)-vanishing in Ω if it satisfies

$$\sup_{0<r\leq R}\sup_{x_0\in\Omega} ⨍_{B_r(x_0)\cap\Omega} \beta(\mathbf{a}, B_r(x_0))(x)\,dx \leq \delta, \tag{3.9}$$

where

$$\beta(\mathbf{a}, B_r(x_0))(x) := \sup_{\xi\in\mathbb{R}^n} \frac{|\mathbf{a}(x,\xi) - \overline{\mathbf{a}}_{B_r(x_0)}(\xi))|}{(1+|\xi|)^{p-1}}, \quad \overline{\mathbf{a}}_{B_r(x_0)}(\xi) = ⨍_{B_r(x_0)} \mathbf{a}(x,\xi)dx.$$

We now introduce the definition of quasiconvex domains.

Definition 3.1.3. (Quasiconvex Domain)
A bounded domain Ω is said to be (δ, σ, R)-*quasiconvex* if for all $x \in \partial\Omega$, $0 < r \leq R$, the following properties hold true:

1. there exists a ball $B_{\sigma r}(x_0) \subset \Omega_r(x)$, where x_0 is relative to x and $\sigma \in \left(0, \frac{1}{4}\right)$ is a uniform constant,
2. there exist a hyperplane $A(x, r)$ containing x, a unit normal vector $\vec{n}(x, r)$ to $A(x, r)$, and a half space $H(x, r) = \{y + t\vec{n}(x, r) : y \in L(x, r),\, t \geq -\delta r\}$ such that

$$\Omega_r(x) \subset H(x, r) \cap B_r(x).$$

Remark 3.1.4. Through this paper, there are some notes about δ, σ, R of quasiconvexity.

1. The positive number σ is arbitrary given so that it is independent of the solutions.

2. δ is to be chosen later in the range $\left(0, \frac{1}{2^{n+1}}\right)$

3. By scaling the problem (3.1), one can take $R = 1$ or any number bigger than 1, while δ is invariant under such scaling, see Lemma 3.2.6.

We refer the readers to the section 3 for a further discussion regarding quasiconvex domains.

Now we states the main theorem.

Theorem 3.1.5. *Assume* $1 < p < q < \infty$ *and* $0 < \sigma < \frac{1}{4}$. *Let* $u \in W_0^{1,p}(\Omega)$ *be the weak solution to the equation (3.1) with (3.2), (3.3) and* $\mathbf{f} \in L^q(\Omega, \mathbb{R}^n)$. *Then there exists a small* $\delta = \delta(\sigma, n, p, q, \lambda, \Lambda) > 0$ *such that if* $\mathbf{a}$ *is* (δ, R)*-vanishing and* Ω *is* (δ, σ, R)*-quasiconvex, then* $Du \in L^q(\Omega, \mathbb{R}^n)$ *with the estimate (3.8).*

Our result generalizes the existing regularity results in [11, 17, 19] on nonsmooth domains to more general domains, as a (δ, R)-Reifenberg domain is a special case of (δ, σ, R)-quasiconvex domain when σ is taken in the range $\left(0, \frac{1-\delta}{2}\right)$. We remark that in the last section we discuss how one can extend the regularity result in Theorem 3.1.5 to the setting of Orlicz spaces.

3.2 Notation and preliminary results

We use the following notations.

1. $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$, and $B_r = B_r(0)$.

2. $\Omega_r(x) = \Omega \cap B_r(x)$, and $\Omega_r = \Omega_r(0)$.

3. $\partial\Omega$ is the boundary of the domain Ω, $\partial_w\Omega_r(x) = \partial\Omega \cap B_r(x)$ and $\partial_c\Omega_r(x) = \partial\Omega_r(x) \setminus \partial_w\Omega_r(x)$.

4. $|E|$ denotes the Lebesgue measure of the set $E \subset \mathbb{R}^n$.

5. $\overline{u}_E = \fint_E u(x)dx = \frac{1}{|E|}\int_E u(x)dx$ is the integral average of u over E.

6. $D(E,F) = \max\left\{\sup_{x\in E} dist(x,F), \sup_{y\in F} dist(y,E)\right\}$ denotes the Hausdorff distance between two sets E and F in $\mathbb{R}^n$.

Lemma 3.2.1. *[68] Suppose that f is a non-negative measurable function in a bounded domain Ω. Then, for $1 < p < \infty$,*

$$f \in L^p \quad \textit{if and only if} \quad S = \sum_{k\geq 1} m^{kp}\left|\{x \in \Omega : f(x) > \theta m^k\}\right| < \infty$$

for some constants $\theta > 0$ and $m > 1$.

Moreover, we have

$$c^{-1}S \leq \|f\|^p_{L^p(\Omega)} \leq c(S + |\Omega|),$$

where $c = c(\theta, m, p)$ is a positive constant.

We use the famous Hardy-Littlewood maximal function, which allows us to control the local behavior of a function in a scaling invariant way for qualitative study of L^p functions.

Definition 3.2.2. Let f be a locally integrable function on $\mathbb{R}^n$. Then the function

$$(\mathcal{M}f)(x) = \sup_{r>0} \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y)|dy,$$

is called the Hardy-Littlewood maximal function of f.

Further, for a function defined on a bounded $U \subset \mathbb{R}^n$, we can define the Hardy-Littlewood maximal function locally by

$$\mathcal{M}_U f = \mathcal{M}\left(f\chi_U\right)$$

where χ is the standard characteristic function on U.

In the following lemma, we observe basic properties of the Hardy-Littlewood maximal function.

Lemma 3.2.3. *[68]*

1. *(weak 1-1 estimate). If $f \in L^1(\mathbb{R}^n)$, then there is a constant $c = c(n) > 0$ such that, for all $t > 0$,*

$$|\{x \in \mathbb{R}^n : (\mathcal{M}f)(x) > t\}| \leq \frac{c}{t} \int |f(x)| dx. \tag{3.10}$$

2. *(strong p-p estimate). If $f \in L^p(\mathbb{R}^n)$ with $1 < p \leq \infty$, then $\mathcal{M}f \in L^p(\mathbb{R}^n)$ and there is a constant $c = c(n, p) > 0$ such that*

$$\frac{1}{c}\|f\|_{L^p} \leq \|\mathcal{M}f\|_{L^p} \leq c\|f\|_{L^p}. \tag{3.11}$$

The weak 1-1 estimate asserts that $\{x \in \mathbb{R}^n : (\mathcal{M}f)(x) > t\}$ and $\{x \in \mathbb{R}^n : |f|(x) > t\}$ have roughly the same measure. Also, the strong p-p estimate claims that $|\{x \in \mathbb{R}^n : |f(x)| > t\}|$ and $|\{x \in \mathbb{R}^n : (\mathcal{M}f)(x) > t\}|$ decay in the same way in L^p sense. However $(\mathcal{M}f)$ behaves much more, as it has a scaling invariant property.

We next state the following Vitali covering lemma, which is one of our main tools.

Lemma 3.2.4. *[68] Let $\{B_\alpha\}$ be any collection of balls in $\mathbb{R}^n$. Then there exists a countable subcollection $\{B_{\alpha_i}\}$ of balls which are disjoint and*

$$\bigcup_\alpha B_\alpha \subset \bigcup_i 5B_{\alpha_i},$$

where $5B_{\alpha_i}$ denotes the ball with the same center as B_{α_i} but with five times the radius.

We apply the following covering lemmas to prove our global regularity estimates. These modified covering lemmas accommodates the special needs for the conditions of small BMO semi-norm and quasiconvex domains.

Lemma 3.2.5. *[48] Assume that C and D are measurable sets, $C \subset D \subset \Omega$ with Ω $(\delta, \sigma, 1)$-quasiconvex, and that there exists an $\epsilon > 0$ such that $|C| < \epsilon|B_1|$ and that for all $x \in B_1$ and for all $r \in (0, 1]$ with $|C \cap B_r(x)| \geq \epsilon|B_r(x)|$, $B_r(x) \cap \Omega \subset D$. Then,*

$$|C| \leq \left(\frac{5}{\sigma}\right)^n \epsilon|D|.$$

Another tool that makes our argument clean is scaling and normalization. Consider the following scaled and normalized setting: for $0 < r < 1$ and $\lambda > 1$,

$$\tilde{\mathbf{a}}(x,\xi) = \frac{\mathbf{a}(rx,\lambda\xi)}{\lambda^{p-1}}, \quad \tilde{u}(x) = \frac{u(rx)}{\lambda r}, \quad \tilde{f}(x) = \frac{f(rx)}{\lambda}, \quad \tilde{\Omega} = \frac{1}{r}\Omega.$$

Then there holds.

Lemma 3.2.6. *1. If u is the weak solution of (3.1), then $\tilde{u}$ is the weak solution of*

$$\begin{cases} div\,\tilde{\mathbf{a}}(x, D\tilde{u}) = div\,(|\tilde{f}|^{p-2}\tilde{f}) & in \quad \tilde{\Omega} \\ \tilde{u} = 0 & on \quad \partial\tilde{\Omega}. \end{cases}$$

2. *If $\mathbf{a}$ satisfies the assumptions (3.2) and (3.3), then so does $\tilde{\mathbf{a}}$ with the same constants λ and Λ.*

3. *If $\mathbf{a}$ is (δ, R)-vanishing in Ω, then $\tilde{\mathbf{a}}$ is $\left(\delta, \frac{R}{r}\right)$-vanishing in $\tilde{\Omega}$.*

4. *If Ω is (δ, σ, R)-quasiconvex, then $\tilde{\Omega}$ is $\left(\delta, \sigma, \frac{R}{r}\right)$-quasiconvex.*

Proof. The proof follows from a direct computation. □

3.3 Quasiconvex Domains

Recently, Jia, Li, and Wang brought up the concept of quasiconvex domains into the regularity theory in their papers [48, 49, 50]. Notice that (δ, σ, R)-quasiconvexity is defined in Definition 3.1.3, where one can see that the notion of quasiconvexity is a generalization of the Reifenberg flatness. In this section, we study basic properties of quasiconvex domains. Roughly speaking, the boundary of a quasiconvex domain, in all scales, can be approximated from inside and outside by two convex surfaces, rather than two hyperplanes for Reifenberg flat domains.

Lemma 3.3.1. *Suppose that Ω is a (δ, σ, R)-quasiconvex domain. Then for any $x \in \partial\Omega$ and $r \in (0, R/2]$, there exists a convex domain $F(x, r)$ such that*

$$\Omega_r(x) \subset F(x, r) \cap B_r(x)$$

and

$$D\left[\partial_w(F(x,r)\cap B_r(x)), \partial_w\Omega_r(x)\right] \le c\delta r,$$

where $c = c(\sigma)$.

Proof. For each $x \in \partial\Omega$ and $r \in \left(0, \frac{R}{2}\right]$, define

$$F(x,r) := \bigcap_{y\in\partial_w\Omega_r(x)} H(y,2r),$$

where $H(y,2r)$ is defined in Definition 3.1.3. Since $F(x,r)$ is an intersection of upper half spaces, it is clear that $F(x,r)$ is convex. We now claim that

$$\Omega_r(x) \subset F(x,r)\cap B_r(x). \tag{3.12}$$

To see this, observe that $\Omega_{2r}(y) \subset H(y,2r)\cap B_{2r}(y)$ for all $y \in \partial\Omega$ since $2r \in (0,R]$. This implies that

$$\Omega_r(x) \subset H(y,2r),\ \forall y \in \partial_w\Omega_r(x)$$

since $\Omega_r(x) \subset \Omega_{2r}(y)$ for all $y \in \partial_w\Omega_r(x)$. This is equivalent to

$$\Omega_r(x) \subset \bigcap_{y\in\partial_w\Omega_r(x)} H(y,2r) = F(x,r),$$

which proves the claim (3.12).

We next fix a point $z \in \partial F(x,r)\cap B_r(x)$. By the definition (3.3), there exists a point x_0 such that $B_{\sigma r}(x_0) \subset \Omega_r(x)$. Let $L(z,x_0)$ be the half line from x_0 to z, and $y \in \partial\Omega\cap L(x_0,z)$ be the closest point to z. Take θ as the angle of $L(z,x_0)$ and $A(y,2r)$ and denote the intersection point of $L(z,x_0)$ and $A(y,2r)$ by z_1. Then

$$dist(z,\partial_w\Omega_r(x)) \le dist(y,z_1) = \frac{2r\delta}{\sin\theta}.$$

Where the inequality holds since $\partial H(y,2r)$ lies below $\partial F(x,r)$, while the equality holds since $A(y,2r)$ is parallel to $\partial H(y,2r)$. Considering a simple geometric picture, we know that

$$\sin\theta > \frac{(\sigma+\delta)r}{\sqrt{(2r)^2+(\sigma+\delta)^2r^2}} > \frac{\sigma}{\sqrt{4+(\sigma+\delta)^2}} \ge \frac{\sigma}{\sqrt{4+(1/2+1)^2}} > \frac{\sigma}{3}.$$

Therefore, we have

$$\sup\{dist(a, \partial_w \Omega_r(x)) : a \in \partial_w(F(x,r) \cap B_r(x))\} \le \frac{6}{\sigma}\delta r,$$

and

$$\sup\{dist(b, \partial_w(F(x,r) \cap B_r(x))) : b \in \partial_w \Omega_r(x)\} = 0$$

since $\Omega_r(x) \subset F(x,r) \cap B_r(x)$. Combining these, we finally conclude that

$$D\left[\partial_w(F(x,r) \cap B_r(x)), \partial_w \Omega_r(x)\right] \le \frac{6}{\sigma}\delta r.$$

□

Lemma 3.3.2. *For the convex domain $F(x,r)$ constructed in Lemma 3.3.1, there exists a convex domain $F^*(x,r)$ such that*

$$F_r^*(x) := F^*(x,r) \cap B_r(x) \subset \Omega_r(x) \text{ and } D[\partial_w F_r^*(x), \partial_w \Omega_r(x)] \le \frac{32\delta r}{\sigma^3}.$$

Proof. Using the fact that there exists a point $x_0 \in \Omega_r(x)$ such that $B_{\sigma r}(x_0) \subset \Omega_r(x)$, we can use the n-dimensional spherical coordinate system centered at x_0. Define

$$F^*(x,r) := \left\{(\rho, \theta_1, \ldots, \theta_{n-1}) | \rho \le \rho'\Big(1 - \frac{16\delta}{\sigma^3}\Big),\ (\rho', \theta_1, \ldots, \theta_{n-1}) \in \partial F(x,r)\right\}. \tag{3.13}$$

For any $y \in F_r^*(x)$, we denote by $L(x_0, y)$ the straight line passing through x_0 and y. Let $y_0 \in \partial_w F(x,r)$ be the intersection point of $\partial_w F(x,r)$ and $L(x_0, y)$. Inside $F(x,r)$, there exists a cone which is tangent to the ball $B_{\sigma r}(x_0)$, with its axis at $L(x_0, y)$ and apex at y_0. Let $z_0 \in \partial B_{\sigma r}(x_0)$ be a touching point with the cone, and $x \in L(y_0, z_0)$ be the point such that $L(y,z) \perp L(y_0, z_0)$. Then

$$\frac{|y - z|}{|x_0 - z_0|} = \frac{|y_0 - y|}{|y_0 - x_0|} \ge \frac{16\delta}{\sigma^3}$$

by the ratio of similarity of triangles and (3.13). Observing $|x_0 - z_0| = \sigma r$ and $B_{|y-z|}(y) \subset$ the cone $\subset F_r(x)$, we have

$$dist(y, \partial_w F_r(x)) \ge |y - z| \ge \frac{16\delta}{\sigma^3}|x_0 - z_0| = \frac{16\delta}{\sigma^2} r. \tag{3.14}$$

By the definition of $F(x,r) = \cap_{y\in\partial_w\Omega_r(x)} H(y,2r)$, $\partial_w\Omega_r(x)$ is in the (upper) $2\delta r$-tubular neighborhood of $\partial F(x,r)$. In addition, it is clear that $2\delta r \leq \frac{16\delta r}{\sigma^2}$, hence $F_r^*(x) \subset \Omega_r(x)$ by (3.14). On the other hand we have

$$dist(y, \partial_w\Omega_r(x)) \leq \frac{16\delta\rho'}{\sigma^3} \leq \frac{32\delta r}{\sigma^3},\ \forall y \in \partial_w F_r^*(x),$$

by (3.13). □

It is clear that a (δ, R)-Reifenberg flat domain is (δ, σ, R)-quasiconvex for $\sigma = \frac{1-\delta}{2}$ and that an equilateral triangle is $(\delta, \frac{1}{3}, R)$-quasiconvex for all $\delta > 0$. Moreover these domains are $W^{1,p}$ extension domains (see [44]). Hence, the extension theorem and Sobolev embedding theorem are available on these domains. Also, the property (2) in the definition implies that quasiconvex domains are locally and approximately convex domains in the following sense.

Lemma 3.3.3. *If Ω is a (δ, σ, R)-quasiconvex domain, then for each $x \in \partial\Omega$ and for every $r \in \left(0, \frac{R}{2}\right)$, there exist two convex domains $F_r(x)$ and $F_r^*(x)$ such that*

$$F_r^*(x) \subset \Omega_r(x) \subset F_r(x) \quad \textit{and} \quad D\left(F_r(x), F_r^*(x)\right) \leq \frac{34\delta r}{\sigma^3}.$$

It is worthwhile to note that

$$F_r(x) = \bigcap_{y\in\partial_w\Omega_r(x)} H(y,2r) \cap B_r(x) \quad \text{and}$$

$$F_r^*(x) = \left\{ x_0 + \left(1 - \frac{16r\delta}{\sigma^3}\right)(y - x_0) : y \in F_r(x) \right\},$$

where $H(y,2r)$ and $x_0 \in \Omega_r(x)$ are given in Definition 3.1.3.

3.4 Proof of the Main Theorem

The main point in this paper is to extend the interior $W^{1,q}$-estimate, $q \geq p$, in [18] to the global estimate of the quasiconvex domains. To do this, we first need the following lemma from [18] which is an interior version of Lemma 3.4.5.

Lemma 3.4.1. *[18] Let $B_{6r}(y) \subset \Omega$ for $0 < r \leq 1$. Assume that $u \in W^{1,p}(\Omega)$ is a weak solution of (3.1) in Ω. Then there is a constant $N_1 = N_1(\lambda, \Lambda, n, p) > 1$ so that for any $0 < \epsilon < 1$, one can find a small constant $\delta = \delta(\epsilon) > 0$ such that if $\mathbf{a}$ satisfies (3.3) and $(\delta, 6)$-vanishing condition and*

$$|\{x \in B_r(y) : \mathcal{M}\left(|Du|^p\right)(x) > N_1^p\}| \geq \epsilon |B_r(y)|,$$

then we have

$$B_r(y) \subset \{x \in \Omega : \mathcal{M}\left(|Du|^p\right)(x) > 1\} \cup \{x \in \Omega : \mathcal{M}\left(|\mathbf{f}|^p\right)(x) > \delta^p\}.$$

We now conduct the approximation estimates near the boundary in the process of deriving the global decay estimates. Consider a localized problem and its corresponding reference problem:

$$\begin{cases} \operatorname{div} \mathbf{a}(x, Du) = \operatorname{div}\left(|\mathbf{f}|^{p-2}\mathbf{f}\right) & \text{in} \quad \Omega_6, \\ u = 0 & \text{on} \quad \partial_w \Omega_6, \end{cases} \tag{3.15}$$

and

$$\begin{cases} \operatorname{div} \bar{\mathbf{a}}_{F_6^*}(Dv) = 0 & \text{in} \quad F_6^*, \\ v = 0 & \text{on} \quad \partial_w F_6^*, \end{cases} \tag{3.16}$$

where

$$F_6^* \subset \Omega_6 \subset F_6 \text{ and } D(F_6, F_6^*) \leq \frac{204}{\sigma^3}\delta, \tag{3.17}$$

by Lemma 3.3.3. The lemma below provides the Lipschitz regularity for the limiting equation (3.16).

Lemma 3.4.2. *Suppose that v is a weak solution of (3.16) under the assumptions (3.2) and (3.3). Then there holds*

$$\|Dv\|^p_{L^\infty(B_2 \cap F_6^*)} \leq c\left(\fint_{F_6^*} |Dv|^p dx + 1\right), \tag{3.18}$$

provided that δ is sufficiently small.

Proof. In this proof, let us denote $\bar{\mathbf{a}}_{F_6^*}$ by $\mathbf{a}$ for convenience sake. Since it is well known that (3.18) holds true for the interior case, it suffices to consider only the boundary case.

We first consider the singular case, that $1 < p < 2$, we have

$$0 = \operatorname{div} \mathbf{a}(Dv) = \frac{\partial}{\partial x_i}(a^i(Dv)) = a_{ij}(Dv)v_{ji} \text{ in } F_6^*, \tag{3.19}$$

where $\mathbf{a} = (a^1, \cdots, a^n)$, $a_{ij}(\xi) := D_{\xi_j} a^i(\xi)$ and $v_{ji} := v_{x_j x_i}$. Hence

$$b_{ij}v_{ij} := \left(\frac{a_{ji}(Dv)}{|Dv|^{p-2}}\right) v_{ij} = 0 \text{ in } F_6^*. \tag{3.20}$$

And we see that by (3.2) and (3.3)

$$\sum_{i,j=1}^{n} b_{ij}\eta_i\eta_j \geq \lambda|\eta|^2, \text{ and } |b| \leq \Lambda. \tag{3.21}$$

Also we observe that

$$\begin{aligned} b_{ij}|x|^{-\alpha}_{x_i x_j} &= \alpha(\alpha+2)|x|^{-\alpha-4} b_{ij}x_i x_j - \alpha|x|^{-\alpha-2} b_{ij}\delta_{ij} \\ &= \alpha|x|^{-\alpha-2}\left((\alpha+2)|x|^{-2} b_{ij}x_i x_j - b_{ij}\delta_{ij}\right) \\ &\geq \alpha|x|^{-\alpha-2}\left((\alpha+2)\lambda - \Lambda\right) > 0, \end{aligned}$$

provided $\alpha \geq \frac{\Lambda}{\lambda} - 2$. So $|x|^{-\alpha}$ is a subsolution, or $-|x|^{-\alpha}$ is a supersolution.

Let us fix a point $x^0 \in B_2 \cap \partial F_6^*$. Then we can find y_0 such that $B_{\frac{1}{2}}(y^0)$ touches F_6^* at x^0. Since there exists a bounded weak solution of (3.16), see [51], one can define a barrier function $h(x) = \frac{|x-y^0|^{-\alpha}-2^{\alpha}}{1-2^{\alpha}}\|v\|_{L^\infty(B_4\cap F_6^*)}$. Then $h(x) = \|v\|_{L^\infty(B_4\cap F_6^*)}$ on $\partial B_1(y^0)$ and $h(x) = 0$ on $\partial B_{\frac{1}{2}}(y^0)$. Since $v - h \leq 0$ on $\partial(F_6^* \cap B_1(y^0))$, we know $v \leq h$ in $F_6^* \cap B_1(y^0)$ by the maximum principle. Thus we find

$$v(x) \leq h(x) \leq c\|v\|_{L^\infty(B_4\cap F_6^*)}|x - x^0| \text{ for } x \in B_1(y^0) \cap F_6^* \cap L(y^0, x^0), \tag{3.22}$$

where $L(y^0, x^0)$ is the line passing through y^0 and x^0. Conversely, for each $x \in B_2 \cap F_6^*$ with

$$\mathbf{d}(x) := dist(x, \partial_w F_6^*) < \frac{1}{2},$$

we can find $x^0 \in B_2 \cap \partial F_6^*$ such that $\mathbf{d}(x) = |x - x^0|$, so (3.22) holds true.

Consequently, we have

$$v(x) \le C\|v\|_{L^\infty(B_4\cap F_6^*)}\mathbf{d}(x) \text{ for } x \in \left\{x \in F_6^* \cap B_2 | \mathbf{d}(x) < \frac{1}{2}\right\}. \tag{3.23}$$

We next use the local boundedness and Poincaré-Sobolev inequality, we have

$$\|v\|_{L^\infty(B_4\cap F_6^*)} \le c\|v\|_{L^{\frac{np}{n-p}}(B_5\cap F_6^*)} \le c\|Dv\|_{L^p(F_6^*)}. \tag{3.24}$$

If we differentiate (3.19) with respect to x_k, $1 \le k \le n$, then we get

$$0 = \operatorname{div}\left(\frac{\partial}{\partial x_k}\mathbf{a}(Dv)\right) = \partial_i\left(a_{ij}v_{jk}\right) = \partial_i\left(b_{ij}|Dv|^{p-2}v_{jk}\right) \text{ in } F_6^*. \tag{3.25}$$

Multiplying (3.25) by v_k, we have

$$\begin{aligned}0 &= v_k\partial_i\left(b_{ij}|Dv|^{p-2}v_{jk}\right) = \partial_i\left(b_{ij}|Dv|^{p-2}v_{jk}v_k\right) - v_{ki}b_{ij}|Dv|^{p-2}v_{jk}\\ &= \partial_i\left(\frac{1}{p}b_{ij}\frac{\partial}{\partial x_j}|Dv|^p\right) - v_{ki}b_{ij}|Dv|^{p-2}v_{jk}.\end{aligned}$$

Since $v_{ki}b_{ij}|Dv|^{p-2}v_{jk} \ge \lambda|Dv|^{p-2}|Dv_k|^2 \ge 0$ by (3.21), we can see that $|Dv|^p$ is a subsolution of (3.20). Then using the local boundedness, we see that for $B_{2r}(y) \subset F_6^*$

$$\sup_{B_r(y)}|Dv|^p \le \frac{c}{r^n}\int_{B_{2r}(y)}|Dv|^p dx = c\fint_{B_{2r}(y)}|Dv|^p dx. \tag{3.26}$$

We know from Caccioppoli inequality for (3.19) that for each $B_{2r}(y) \subset F_6^*$,

$$\fint_{B_r(y)}|Dv|^p dx \le \frac{c}{r^p}\fint_{B_{2r}(y)}|v|^p dx. \tag{3.27}$$

Combining (3.26), (3.27), (3.23) and (3.24), we find for $B_{4r}(y) \subset \left\{x \in F_6^* \cap B_2 | \mathbf{d}(x) < \frac{1}{2}\right\}$;

$$\begin{aligned}|Dv(y)|^p &\le c\fint_{B_{2r}(y)}|Dv|^p dx \le \frac{c}{r^p}\fint_{B_{4r}(y)}|v|^p dx \\ &\le \frac{c}{r^p}\fint_{B_{4r}(y)}\|v\|^p_{L^\infty(B_4\cap F_6^*)}\mathbf{d}^p(x)dx \le c\|v\|^p_{L^\infty(B_4\cap F_6^*)} \le c\|Dv\|^p_{L^p(F_6^*)}.\end{aligned} \tag{3.28}$$

Passing y to $x^0 \in \partial_w F_6^*$ as $r \to 0$, we get the result (3.18).

We next consider the degenerate case that $p \geq 2$. Define $\mathbf{a}^\epsilon(\xi) = \mathbf{a}(\xi) + \epsilon\xi$. Then, by direct computation using (3.2) and (3.3), we have

$$|\mathbf{a}^\epsilon(\xi)| + |\xi||D\mathbf{a}^\epsilon(\xi)| \leq \Lambda|\xi|^{p-1} + \epsilon|\xi| \leq 2\Lambda(|\xi|^{p-1} + 1) \text{ and}$$

$$\langle D_\xi \mathbf{a}^\epsilon(\xi)\eta,\, \eta\rangle = \langle D_\xi \mathbf{a}(\xi)\eta,\, \eta\rangle + \epsilon|\eta|^2 \geq \lambda(|\xi|^{p-2} + \epsilon)|\eta|^2 \geq \lambda|\xi|^{p-2}|\eta|^2.$$

Then one can find a function $v^\epsilon \in W^{1,p}(F_6^*)$ to satisfy

$$\begin{cases} \operatorname{div} \mathbf{a}^\epsilon(Dv) = 0 & \text{in} \quad F_6^*, \\ \quad v = 0 & \text{on} \quad \partial_w F_6^*, \end{cases} \tag{3.29}$$

in the weak sense. And then, we observe

$$0 = \operatorname{div} \mathbf{a}^\epsilon(Dv^\epsilon) = \frac{\partial}{\partial x_i}\left((a^\epsilon)^i(Dv^\epsilon)\right) = a_{ij}^\epsilon(Dv^\epsilon)v_{ji}^\epsilon \text{ in } F_6^*, \tag{3.30}$$

where $\mathbf{a}^\epsilon = ((a^\epsilon)^1, \cdots, (a^\epsilon)^n)$, $a_{ij}^\epsilon(\xi) := D_{\xi_j}(a^\epsilon)^i(\xi)$ and $v_{ji}^\epsilon := v_{x_j x_i}^\epsilon$. So we can see

$$b_{ij}^\epsilon v_{ij}^\epsilon := \left(\frac{a_{ji}^\epsilon(Dv^\epsilon)}{(|Dv^\epsilon| + \epsilon)^{p-2}}\right) v_{ij}^\epsilon = 0 \text{ in } F_6^*, \tag{3.31}$$

And we have ellipticity and boundedness of b^ϵ;

$$\sum_{i,j=1}^n b_{ij}^\epsilon \eta_i\eta_j = \sum_{i,j=1}^n \frac{\left(D_{\xi_j}a^i(Dv^\epsilon) + \epsilon\right)\eta_i\eta_j}{(|Dv^\epsilon| + \epsilon)^{p-2}} \geq \frac{\lambda|Dv^\epsilon|^{p-2} + \epsilon}{(|Dv^\epsilon| + \epsilon)^{p-2}}|\eta|^2 \geq c_p\lambda|\eta|^2,$$

and

$$|b^\epsilon| \leq \frac{\Lambda|Dv^\epsilon|^{p-2} + \epsilon}{(|Dv^\epsilon| + \epsilon)^{p-2}} \leq c_p\Lambda.$$

Then we have

$$b_{ij}^\epsilon |x|_{x_i x_j}^{-\alpha} > 0,$$

for some α, as in the singular case. So $|x|^{-\alpha}$ is a subsolution of b^ϵ, too. And then we can see

$$v^\epsilon(x) \leq C\|v^\epsilon\|_{L^\infty(B_4 \cap F_6^*)}\mathbf{d}(x) \text{ for } x \in \left\{x \in F_6^* \cap B_2 | \mathbf{d}(x) < \frac{1}{2}\right\},$$

like (3.23), through the same process with the singular case. If we differentiate

(3.30) with respect to x_k, $1 \le k \le n$, then we get

$$\begin{aligned} 0 &= \operatorname{div}\left(\tfrac{\partial}{\partial x_k}\mathbf{a}(Dv^\epsilon) + \epsilon Dv^\epsilon\right) = \partial_i\left(a_{ij}(Dv^\epsilon)v^\epsilon_{jk} + \epsilon\delta_{ij}v^\epsilon_{ik}\right) \\ &= \partial_i\left(b^\epsilon_{ij}(|Dv^\epsilon| + \epsilon)^{p-2}v_{jk}\right) \text{ in } F^*_6. \end{aligned} \tag{3.32}$$

Multiplying (3.32) by v^ϵ_k, we have

$$\begin{aligned} 0 &= v^\epsilon_k\partial_i\left(b^\epsilon_{ij}(|Dv^\epsilon| + \epsilon)^{p-2}v^\epsilon_{jk}\right) \\ &= \partial_i\left(b^\epsilon_{ij}(|Dv^\epsilon| + \epsilon)^{p-2}v^\epsilon_{jk}v^\epsilon_k\right) - v^\epsilon_{ki}b^\epsilon_{ij}(|Dv^\epsilon| + \epsilon)^{p-2}v^\epsilon_{jk} \\ &= \partial_i\left(\frac{1}{p}b_{ij}\frac{\partial}{\partial x_j}(|Dv^\epsilon| + \epsilon)^p\right) - v^\epsilon_{ki}b^\epsilon_{ij}(|Dv^\epsilon| + \epsilon)^{p-2}v^\epsilon_{jk}. \end{aligned}$$

Since $v^\epsilon_{ki}b^\epsilon_{ij}|Dv^\epsilon|^{p-2}v^\epsilon_{jk} \ge c_p\lambda|Dv^\epsilon|^{p-2}|Dv^\epsilon_k|^2 \ge 0$ by (3.21), we can see that $(|Dv^\epsilon| + \epsilon)^p$ is a subsolution of (3.31). And then, we have

$$|Dv^\epsilon(y)|^p \le C\left(1 + \|Dv^\epsilon\|^p_{L^p(F^*_6)}\right) \text{ for } B_{4r}(y) \subset \left\{x \in F^*_6 \cap B_2 | \mathbf{d}(x) < \frac{1}{2}\right\}. \tag{3.33}$$

like (3.28), through the same process with singular case. Let $\eta \in C^\infty_0(F^*_6)$ with $0 \le \eta \le 1$, and $0 < \epsilon < \frac{\tilde{\lambda}}{2p}$.

$$\begin{aligned} \int_{F^*_6} |Dv^\epsilon - Dv|^p\eta dx &\le \frac{1}{\tilde{\lambda}}\int_{F^*_6}\langle\mathbf{a}^\epsilon(Dv^\epsilon) - \mathbf{a}^\epsilon(Dv), Dv^\epsilon - Dv\rangle\eta dx \\ &= \frac{1}{\tilde{\lambda}}\left(\int_{F^*_6}\langle\mathbf{a}^\epsilon(Dv), Dv - Dv^\epsilon\rangle\eta dx + \int_{F^*_6}\langle\mathbf{a}^\epsilon(Dv^\epsilon), Dv - Dv^\epsilon\rangle\eta dx\right) \\ &\overset{(3.29)}{=} \frac{1}{\tilde{\lambda}}\int_{F^*_6}\langle\mathbf{a}^\epsilon(Dv), Dv - Dv^\epsilon\rangle\eta dx \\ &= \frac{\epsilon}{\tilde{\lambda}}\int_{F^*_6}\langle Dv, Dv - Dv^\epsilon\rangle\eta dx \\ &\le \frac{\epsilon(p-1)}{\tilde{\lambda}p}\int_{F^*_6}|Dv|^{\frac{p}{p-1}}\eta dx + \frac{\epsilon}{\tilde{\lambda}p}\int_{F^*_6}|Dv - Dv^\epsilon|^p\eta dx. \end{aligned}$$

Since $0 < \epsilon < \frac{\tilde{\lambda}}{2p}$, we have

$$\int_{F_6^*} |Dv^\epsilon - Dv|^p \eta dx \leq \epsilon c \|Dv\|_{L^p(F_6^*)}^{\frac{p-1}{p}},$$

where $c = c(p, \tilde{\lambda})$. Since $\eta \in C_0^\infty(F_6^*)$ is arbitrary, $Dv^\epsilon \to Dv$ in $L^p(F_6^*)$ as $\epsilon \to 0$. So (3.18) follows by (3.33) by letting $\epsilon \to 0$. □

The following lemma is the first step for our approximation argument considering the quasiconvexity of the domain.

Lemma 3.4.3. *Assume that* $\mathbf{a}$ *satisfies (3.2), (3.3). Then, given* $\epsilon > 0$*, there exists* $\delta = \delta(\epsilon, \lambda, \Lambda, n, p, \sigma) > 0$ *so that if for any weak solution* $u \in W^{1,p}(\Omega_6)$ *of (3.15) with*

$$\fint_{\Omega_6} |Du|^p dx \leq 1, \tag{3.34}$$

$$\fint_{\Omega_6} (|\beta(\mathbf{a}, \Omega_6)|^p + |\mathbf{f}|^p) dx \leq \delta^p \tag{3.35}$$

and (3.17) hold, then there exists a weak solution $v \in W^{1,p}(F_6^*)$ *of (3.16) with*

$$\fint_{F_6^*} |Dv|^p dx \leq 1, \tag{3.36}$$

such that

$$\fint_{F_6^*} |u - v|^p dx \leq \epsilon^p. \tag{3.37}$$

Proof. If not, there exist $\epsilon_0 > 0$, $\{\mathbf{a}_k\}_{k=1}^\infty$, $\{(F_6^k, \Omega_6^k, (F_6^k)^*)\}$ and $\{\mathbf{f}_k\}_{k=1}^\infty$ such that $u_k \in W^{1,p}(\Omega_4^k)$ is a weak solution of

$$\begin{cases} \operatorname{div} \mathbf{a}_k(x, Du_k) = \operatorname{div}(|\mathbf{f_k}|^{p-2}\mathbf{f_k}) & \text{in} \quad \Omega_6^k, \\ u_k = 0 & \text{on} \quad \partial_w \Omega_6^k, \end{cases} \tag{3.38}$$

where

$$(F_6^k)^* \subset \Omega_6^k \subset F_6^k, \quad D(F_6^k, (F_6^k)^*) \leq \frac{204}{\sigma^3 k}, \quad B_{6\sigma}(x_0^k) \subset \Omega_6^k, \tag{3.39}$$

and

$$\fint_{\Omega_6^k} |Du_k|^p dx \leq 1, \quad \fint_{\Omega_6^k} (|\beta\left(\mathbf{a}_k, \Omega_6^k\right)|^p + |\mathbf{f_k}|^p) dx \leq 1/k^p, \tag{3.40}$$

but

$$\fint_{(F_6^k)^*} |(u_k - v)|^p dx > \epsilon_0^p \tag{3.41}$$

for any weak solution v of

$$\begin{cases} \operatorname{div} \overline{\mathbf{a}}_{(F_6^k)^*}(Dv) = 0 & \text{in} \quad (F_6^k)^*, \\ v = 0 & \text{on} \quad \partial_w (F_6^k)^*, \end{cases} \tag{3.42}$$

with

$$\fint_{(F_6^k)^*} |Dv|^p dx \leq 1.$$

Considering the inside ball condition in (3.39), passing to a subsequence if necessary, we may assume that there is a uniform ball, say $B_{6\sigma}(x_0)$, inside each Ω_6^k with $|x_0 - x_0^k| < \frac{1}{k}$ for sufficiently large k. Then, since F_6^k and $(F_6^k)^*$ are convex with $D(F_6^k, (F_6^k)^*) \leq \frac{c'}{k}$, where $c' = \frac{204}{\sigma^3}$, $\{F_6^k\}$ and $\{(F_6^k)^*\}$ are compact in Hausdorff convergence. Hence, they have subsequences convergent to a convex set F_6, and so does $\{\Omega_6^k\}$. Passing to another subsequence, we additionally assume that $(F_6^k)^* \subset F \subset F_6^k$ for all $k \geq 1$. Since $u_k = 0$ on $\partial_w \Omega_6^k$, we can assume that $u_k = 0$ in $B_6 \setminus \Omega_6^k$ by the zero extension. Then it follows from (3.39), (3.40) and Poincaré-Sobolev inequality that

$$\int_F |u_k|^p dx \leq \int_{B_6} |u_k|^p dx \leq C \int_{B_6} |Du_k|^p dx = C \int_{\Omega_6^k} |Du_k|^p dx \leq C.$$

Hence $\{u_k\}$ is uniformly bounded in $W^{1,p}(F)$ and so there exist $u_0 \in W^{1,p}(F)$ and a subsequence, still denoted by $\{u_k\}$, such that

$$\begin{cases} u_k \rightharpoonup u_0 & \text{weakly in} \quad W^{1,p}(F), \\ u_k \to u_0 & \text{strongly in} \quad L^p(F). \end{cases} \tag{3.43}$$

On the other hand, for each fixed ξ,

$$|\overline{\mathbf{a}_k}_{(F_6^k)^*}(\xi)| \leq \fint_{(F_6^k)^*} |\mathbf{a}_k(x, \xi)| dx \leq \Lambda \fint_{(F_6^k)^*} |\xi|^{p-1} dx = \Lambda |\xi|^{p-1} < \infty.$$

Then, passing to a subsequence, one can find $\mathbf{a}_0$ such that

$$\overline{\mathbf{a}_k}_{(F_6^k)^*}(\xi) \to \mathbf{a}_0(\xi) \text{ in } \mathbb{R}^n, \tag{3.44}$$

for each fixed $\xi \in \mathbb{R}^n$.

Next we assert that u_0 is a weak solution of

$$\begin{cases} \operatorname{div} \mathbf{a}_0(Du_0) = 0 & \text{in} \quad F, \\ u_0 = 0 & \text{on} \quad \partial_w F. \end{cases} \tag{3.45}$$

To do this, we fix any test function $\varphi \in W_0^{1,p}(F)$ and define $\varphi_k(x) = \bar{\varphi}\left(\frac{x-x_0}{1-c'/k} + x_0\right)$, where $\bar{\varphi}$ is the zero extension of φ. Then $\varphi_k \in W_0^{1,p}\left((F_6^k)^*\right) \subset W_0^{1,p}(\Omega_6^k)$ and $\varphi_k \to \varphi$ in $W_0^{1,p}(F)$. Hence it follows from (3.38) that

$$\int_{\Omega_6^k} \langle \mathbf{a}_k(x, Du_k), D\varphi_k \rangle \, dx = \int_{\Omega_6^k} \left\langle |\mathbf{f}_k|^{p-2}\mathbf{f}_k, D\varphi_k \right\rangle dx. \tag{3.46}$$

Furthermore, considering $u_k = 0$ in $F \setminus \Omega_6^k$ by the zero extension, and using the growth condition (3.3) and (3.40), we have

$$\int_F |\mathbf{a}_k(x, Du_k)|^{\frac{p}{p-1}} dx \le \Lambda^{\frac{p}{p-1}} |\Omega_6^k| \fint_{\Omega_6^k} |Du_k|^p dx \le C.$$

Thus, there exists a vector-valued function $\mathbf{b} \in L^{\frac{p}{p-1}}(F;\mathbb{R}^n)$ and a subsequence of $\{\mathbf{a}_k(x, Du_k)\}$, still written by the same, such that

$$\mathbf{a}_k(x, Du_k) \rightharpoonup \mathbf{b} \text{ in } L^{\frac{p}{p-1}}(F;\mathbb{R}^n). \tag{3.47}$$

Since

$$\begin{aligned} \int_F \langle \mathbf{a}_k(x, Du_k), D\varphi_k \rangle \, dx = & \int_F \langle \mathbf{a}_k(x, Du_k), D\varphi \rangle \, dx \\ & - \int_F \langle \mathbf{a}_k(x, Du_k), D\varphi - D\varphi_k \rangle \, dx, \end{aligned}$$

and

$$\int_F \langle \mathbf{a}_k(x, Du_k), D\varphi - D\varphi_k \rangle \, dx \le \|\mathbf{a}_k(x, Du_k)\|_{L^{\frac{p}{p-1}}(F)} \|D\varphi - D\varphi_k\|_{L^p(F)},$$

it follows from (3.35) and (3.47) that

$$\int_F \langle \mathbf{b}, D\varphi \rangle \, dx = 0, \tag{3.48}$$

taking the limit in (3.46). Recalling $u_k = 0$ on $\partial_w \Omega_6$, using (3.39) and (3.43) we find that

$$u_0 = 0 \text{ on } \partial_w F \tag{3.49}$$

in the trace sense. Then owing to (3.48) and (3.49), it remains to show that

$$\mathbf{b} = \mathbf{a_0}(Du_0) \text{ a.e. in } F \tag{3.50}$$

for the assertion (3.45). Let us fix $w \in C^\infty(F)$ and $\phi \in C^\infty(\mathbb{R}^n)$ with $supp(\phi) \subset F$ and $\phi \geq 0$. Define $\phi_k(x) = \phi\left(\frac{x-x_0}{1-\frac{c'}{k}} + x_0\right)$ where $c' = \frac{204}{\sigma^3}$, so that $\phi_k \in C_0^\infty\left((F_6^k)^*\right)$. Then using the monotonicity condition (3.4), we get

$$\begin{aligned}
0 &\leq \int_F \phi_k \langle \mathbf{a_k}(x, Du_k) - \mathbf{a}_k(x, Dw), Du_k - Dw \rangle \, dx \\
&= \underbrace{\int_F \phi_k \langle \mathbf{a}_k(x, Du_k), Du_k \rangle \, dx}_{I_1} - \underbrace{\int_F \phi_k \langle \mathbf{a}_k(x, Du_k), Dw \rangle \, dx}_{I_2} \\
&\quad - \underbrace{\int_F \phi_k \left\langle \mathbf{a}_k(x, Dw) - \overline{\mathbf{a}_k}_{(F_6^k)^*}(Dw), Du_k - Dw \right\rangle dx}_{I_3} \\
&\quad - \underbrace{\int_F \phi_k \left\langle \overline{\mathbf{a}_k}_{(F_6^k)^*}(Dw), Du_k - Dw \right\rangle dx}_{I_4}.
\end{aligned}$$

We take $\phi_k u_k$ for the test function in (3.38) to find

$$
\begin{aligned}
I_1 &= \int_{\Omega_6^k} \phi_k \langle \mathbf{a}_k(x, Du_k), Du_k \rangle \, dx \\
&= -\int_{\Omega_6^k} u_k \langle \mathbf{a}_k(x, Du_k), D\phi_k \rangle \, dx + \int_{\Omega_6^k} \phi_k \left\langle |\mathbf{f}_k|^{p-2}\mathbf{f}_k, Du_k \right\rangle dx \\
&\quad + \int_{\Omega_6^k} u_k \left\langle |\mathbf{f}_k|^{p-2}\mathbf{f}_k, D\phi_k \right\rangle dx.
\end{aligned}
$$

Recalling (3.40), (3.43) and (3.47), and taking $k \to \infty$, we have that

$$
I_1 \to -\int_F u_0 \langle \mathbf{b}, D\phi \rangle \, dx. \tag{3.51}
$$

Next we take $\varphi = \phi u_0$ in the equality (3.48) to find

$$
\int_F \phi \langle \mathbf{b}, Du_0 \rangle \, dx = -\int_F u_0 \langle \mathbf{b}, D\phi \rangle \, dx.
$$

Then (3.51) implies that

$$
I_1 \to \int_F \phi \langle \mathbf{b}, Du_0 \rangle \, dx. \tag{3.52}
$$

We employ (3.47) to find

$$
I_2 = \int_F \phi_k \langle \mathbf{a}_k(x, Du_k), Dw \rangle \, dx \to \int_F \phi \langle \mathbf{b}, Dw \rangle \, dx. \tag{3.53}
$$

Recalling Definition 3.1.2 and $\phi_k \in C_0^\infty\left((F_6^k)^*\right)$, we estimate I_3 as follows:

$$
\begin{aligned}
|I_3| &= \left| \int_{(F_6^k)^*} \phi_k \left\langle \mathbf{a}_k(x, Dw) - \overline{\mathbf{a}_k}_{(F_6^k)^*}(Dw), Du_k - Dw \right\rangle dx \right| \\
&\le \int_{(F_6^k)^*} \phi_k \left[|\mathbf{a}_k(x, Dw) - \overline{\mathbf{a}_k}_{\Omega_6^k}(Dw)| + |\overline{\mathbf{a}_k}_{\Omega_6^k}(Dw) - \overline{\mathbf{a}_k}_{(F_6^k)^*}(Dw)| \right] \\
&\qquad \cdot |Du_k - Dw| dx \\
&\le \int_{(F_6^k)^*} \phi_k \Bigg[\beta(\mathbf{a_k}, \Omega_6^k)(x) \left(1 + |Dw|^{p-1}\right) \\
&\qquad + \fint_{(F_6^k)^*} |\mathbf{a}_k(y, Dw) - \overline{\mathbf{a}_k}_{\Omega_6^k}(Dw)| dy \Bigg] |Du_k - Dw| dx \\
&\le \int_{(F_6^k)^*} \phi_k \left[\beta(\mathbf{a_k}, \Omega_6^k)(x) + \frac{|\Omega_6^k|}{|(F_6^k)^*|} \fint_{\Omega_6^k} \beta(\mathbf{a_k}, \Omega_6^k)(y) dy \right] \left(1 + |Dw|^{p-1}\right) \\
&\qquad \cdot |Du_k - Dw| dx \\
&\le \int_{(F_6^k)^*} \phi_k \left[\beta(\mathbf{a_k}, \Omega_6^k)(x) + \left(1 + \frac{c'}{k}\right)^n \frac{1}{k} \right] \left(1 + |Dw|^{p-1}\right) |Du_k - Dw| dx \\
&\le C \left(\int_{(F_6^k)^*} \left(\phi_k \beta(\mathbf{a_k}, \Omega_6^k)(x)\right)^{\frac{p}{p-1}} (1 + |Dw|^p) \, dx \right)^{\frac{p-1}{p}} \|Du_k - Dw\|_{L^p(F)} \\
&\quad + C \left(1 + \frac{c'}{k}\right)^n \frac{1}{k} \left(\int_F (1 + |Dw|^p) \, dx \right)^{\frac{p-1}{p}} \|Du_k - Dw\|_{L^p(F)} \\
&\le C \left(|\Omega_6^k| \underbrace{\fint_{\Omega_6^k} \left(\phi_k \beta(\mathbf{a_k}, \Omega_6^k)(x)\right)^{\frac{p}{p-1}} (1 + |Dw|^p) \, dx}_{I_3^1} \right)^{\frac{p-1}{p}} \|Du_k - Dw\|_{L^p(F)} \\
&\quad + C \left(1 + \frac{c'}{k}\right)^n \frac{1}{k} \left(\int_F (1 + |Dw|^p) \, dx \right)^{\frac{p-1}{p}} \|Du_k - Dw\|_{L^p(F)}.
\end{aligned}
$$

To estimate I_3^1, we separate it into two cases. For $p \ge 2$, we observe that by the Hölder inequality and (3.40),

$$
\begin{aligned}
I_3^1 &= \fint_{\Omega_6^k} \left(\beta(\mathbf{a_k}, \Omega_6^k)(x)\right)^{\frac{p}{p-1}} \phi_k^{\frac{p}{p-1}} \left(1+|Dw|^p\right) dx \\
&\le \left(\fint_{\Omega_6^k} \beta(\mathbf{a_k}, \Omega_6^k)(x)^p dx\right)^{\frac{1}{p-1}} \left(\fint_{\Omega_6^k} \phi_k^{\frac{p}{p-2}} \left(1+|Dw|^p\right)^{\frac{p-1}{p-2}} dx\right)^{\frac{p-2}{p-1}} \\
&\le \frac{|F|}{k|\Omega_6^k|} \left(\fint_F \phi_k^{\frac{p}{p-2}} \left(1+|Dw|^p\right)^{\frac{p-1}{p-2}} dx\right)^{\frac{p-1}{p}}.
\end{aligned}
$$

For $1<p<2$, we have by the Hölder inequality, boundedness of β and (3.40),

$$
\begin{aligned}
I_3^1 &= \fint_{\Omega_6^k} \left(\beta(\mathbf{a_k}, \Omega_6^k)(x)\right)^{\frac{1}{p-1}} \beta(\mathbf{a_k}, \Omega_6^k)(x) \phi_k^{\frac{p}{p-1}} \left(1+|Dw|^p\right) dx \\
&\le (2\Lambda)^{\frac{1}{p-1}} \left(\fint_{\Omega_6^k} \left(\beta(\mathbf{a_k}, \Omega_6^k)(x)\right)^p dx\right)^{\frac{1}{p}} \left(\fint_{\Omega_6^k} \phi_k^{\frac{p^2}{(p-1)^2}} \left(1+|Dw|^p\right)^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \\
&\le \frac{c|F|}{k|\Omega_6^k|} \left(\fint_F \phi_k^{\frac{p^2}{(p-1)^2}} \left(1+|Dw|^p\right)^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}},
\end{aligned}
$$

where $c=c(\Lambda,p)$. Consequently,

$$
|I_3| \to 0 \text{ as } k \to \infty. \tag{3.54}
$$

To estimate I_4, we observe that

$$
\begin{aligned}
I_4 &= \int_F \phi_k \left\langle \overline{\mathbf{a}_k}_{(F_6^k)^*}(Dw), Du_k - Dw \right\rangle dx \\
&= \int_F \phi_k \left\langle \overline{\mathbf{a}_k}_{(F_6^k)^*}(Dw) - \mathbf{a}_0(Dw) + \mathbf{a}_0(Dw), Du_k - Dw \right\rangle dx \\
&= \int_F \phi_k \left\langle \overline{\mathbf{a}_k}_{(F_6^k)^*}(Dw) - \mathbf{a}_0(Dw), Du_k - Dw \right\rangle dx \\
&\qquad + \int_F \phi_k \left\langle \mathbf{a}_0(Dw), Du_k - Dw \right\rangle dx.
\end{aligned}
$$

Using (3.43), we obtain that

$$\int_F \phi_k \langle \mathbf{a}_0(Dw), Du_k - Dw \rangle \, dx \to \int_F \phi \langle \mathbf{a}_0(Dw), Du_0 - Dw \rangle \, dx.$$

and by (3.44),

$$\begin{aligned} &\int_F \phi_k \left\langle \overline{\mathbf{a}_k}_{(F_6^k)^*}(Dw) - \mathbf{a}_0(Dw), Du_k - Dw \right\rangle dx \\ &\qquad \le \|\phi_k(\overline{\mathbf{a}_k}_{(F_6^k)^*}(Dw) - \mathbf{a}_0(Dw))\|_{L^{\frac{p}{p-1}}(F)} \|Du_k - Dw\|_{L^p(F)} \to 0 \end{aligned}$$

as k $\to \infty$. Thus

$$I_4 \to \int_F \phi \langle \mathbf{a}_0(Dw), Du_0 - Dw \rangle \, dx. \tag{3.55}$$

Recalling the fact that $I_1 - I_2 - I_3 - I_4 \ge 0$ and combining (3.52), (3.53), (3.54) and (3.55), we have

$$\int_F \phi \langle \mathbf{b} - \mathbf{a}_0(Dw), Du_0 - Dw \rangle \, dx \ge 0. \tag{3.56}$$

By approximation, one can see that (3.56) holds for all $w \in W^{1,p}(F)$. Fix $\psi \in C_0^\infty(F)$ and set $w = u_0 - \gamma\psi$ ($\gamma > 0$) in (3.56). Then we obtain

$$\int_F \phi \langle \mathbf{b} - \mathbf{a}_0(Du_0 - \gamma D\psi), D\psi \rangle \, dx \ge 0.$$

Letting $\gamma \to 0$, we have

$$\int_F \phi \langle \mathbf{b} - \mathbf{a}_0(Du_0), D\psi \rangle \, dx \ge 0.$$

for all $\phi \in C_0^\infty(F)$ with $\phi \ge 0$ and $\psi \in W^{1,p}(F)$. Replacing ψ by $-\psi$, we deduce that

$$\int_F \phi \langle \mathbf{b} - \mathbf{a}_0(Du_0 - D\psi), D\psi \rangle \, dx = 0.$$

Since this equality holds for each $\phi \in C_0^\infty(F)$ with $\phi \ge 0$ and each $\psi \in W^{1,p}(F)$, (3.49) follows. Hence u_0 is indeed a weak solution of (3.45) satisfying (3.36). Therefore, we reach a contradiction to (3.41) considering (3.42), (3.43) and (3.44). □

In (3.16), v is only defined in F_6^*, and so it's necessary to extend v from F_6^* to Ω_6. Since $v = 0$ on $\partial_w F_6^*$ in the trace sense, we assume that v is defined in Ω_6 by the zero extension from now on.

Corollary 3.4.4. *Under the same conditions and conclusion as in Lemma 3.4.3, we further have*

$$\fint_{\Omega_{3\sigma}} |D(u-v)|^p dx \le \epsilon^p. \tag{3.57}$$

Proof. We apply Lemma 3.4.3 with $\eta > 0$ replacing ϵ and $\delta(\eta)$ replacing $\delta(\epsilon)$ respectively, to deduce

$$\fint_{F_6^*} |u-v|^p dx \le \eta^p. \tag{3.58}$$

Since v is extended by the zero from F_6^* to Ω_6, we observe that

$$\begin{cases} \operatorname{div} \bar{\mathbf{a}}_{F_6^*}(Dv) = \frac{\partial g}{\partial x_n} & \text{in} \quad \Omega_{5\sigma}, \\ v = 0 & \text{on} \quad \partial_w \Omega_{5\sigma}, \end{cases} \tag{3.59}$$

in the weak sense, where $\bar{\mathbf{a}}_{F_6^*} = (\bar{a}^1_{F_6^*}, \cdots, \bar{a}^n_{F_6^*})$, $x' = (x_1, \cdots, x_{n-1})$, $\partial_w F_6^* = (x', \gamma(x'))$ and

$$g(x) = \chi_{\{x_n < \gamma(x')\}} \left(\sum_{k=1}^{n-1} D_{x_k}\gamma(x') \bar{a}^k_{F_6^*}\left(Dv(x', \gamma(x'))\right) - \bar{a}^n_{F_6^*}\left(Dv(x', \gamma(x'))\right) \right), \tag{3.60}$$

where χ is standard characteristic function. Since F_6^* is a convex domain, $\gamma(x')$ is a convex function which implies $D_{ee}\gamma(x') > 0$ for any unit vector $e \in \mathbb{R}^{n-1}$. Then by the geometry of nondegenerate ball $B_{6\sigma}(x_0)$, we observe

$$D_{x_k}\gamma(x') \le \frac{6}{\sigma} \quad \text{for a.e. } x' \in B_{5\sigma} \cap \left(\mathbb{R}^{n-1} \times \{0\}\right) \text{ and } 1 \le k \le n-1. \tag{3.61}$$

Note that since $v = 0$ on $\partial_w F_6^*$ in the trace sense, Dv is a.e. same with the zero extension of derivative of v defined only F_6^*. So we have, by lemma

3.4.2 and (3.36),

$$\|Dv\|_{L^\infty(\Omega_{5\sigma})} \leq \|Dv\|_{L^\infty(\Omega_2)} = \|Dv\|_{L^\infty(B_2 \cap F_6^*)} \leq c(\fint_{F_6^*} |Dv|^p dx + 1) \leq c. \tag{3.62}$$

From Definition 1.1 and (3.59), we have

$$\fint_{\Omega_{5\sigma}} \langle \mathbf{a}(x, Du), D\phi \rangle \, dx = \fint_{\Omega_{5\sigma}} \left\langle |\mathbf{f}|^{p-2}\mathbf{f}, D\phi \right\rangle dx$$

and

$$\fint_{\Omega_{5\sigma}} \left\langle \bar{\mathbf{a}}_{F_6^*}(Dv), D\phi \right\rangle dx = \fint_{\Omega_{5\sigma}} g\phi_{x_n} dx$$

for all $\phi \in W_0^{1,p}(\Omega_{5\sigma})$. After a simple computation, we have identity

$$\begin{aligned}
&\fint_{\Omega_{5\sigma}} \langle \mathbf{a}(x, Du) - \mathbf{a}(x, Dv), D\phi \rangle \, dx + \fint_{\Omega_{5\sigma}} \left\langle \mathbf{a}(x, Dv) - \bar{\mathbf{a}}_{F_6^*}(Dv), D\phi \right\rangle dx \\
&\quad = \fint_{\Omega_{5\sigma}} \langle |\mathbf{f}|^{p-2}\mathbf{f}, D\phi \rangle \, dx - \fint_{\Omega_{5\sigma}} g\phi_{x_n} dx.
\end{aligned}$$

Now select a cut-off function $\varphi \in C_0^\infty(B_{4\sigma})$ satisfying

$$0 \leq \varphi \leq 1, \ \varphi \equiv 1 \text{ on } B_{3\sigma} \text{ and } |D\varphi| \leq \frac{4}{\sigma}, \tag{3.63}$$

and then substitute $\phi = \varphi^p(u - v)$ into the above identity and write the resulting identity as

$$J_1 = J_2 + J_3 + J_4 + J_5 + J_6 + J_7,$$

for

$$J_1 = \fint_{\Omega_{5\sigma}} \varphi^p \langle \mathbf{a}(x,Du) - \mathbf{a}(x,Dv), Du - Dv\rangle \, dx,$$
$$J_2 = -\fint_{\Omega_{5\sigma}} p\varphi^{p-1}(u-v) \langle \mathbf{a}(x,Du) - \mathbf{a}(x,Dv), D\varphi\rangle \, dx,$$
$$J_3 = -\fint_{\Omega_{5\sigma}} \varphi^p \left\langle \mathbf{a}(x,Dv) - \bar{\mathbf{a}}_{F_6^*}(Dv), Du - Dv\right\rangle dx,$$
$$J_4 = -\fint_{\Omega_{5\sigma}} p\varphi^{p-1}(u-v) \left\langle \mathbf{a}(x,Dv) - \bar{\mathbf{a}}_{F_6^*}(Dv), D\varphi\right\rangle dx,$$
$$J_5 = \fint_{\Omega_{5\sigma}} \varphi^p \left\langle |\mathbf{f}|^{p-2}\mathbf{f}, Du - Dv\right\rangle dx,$$
$$J_6 = \fint_{\Omega_{5\sigma}} p\varphi^{p-1}(u-v) \left\langle |\mathbf{f}|^{p-2}\mathbf{f}, D\varphi\right\rangle dx,$$
$$J_7 = -\fint_{\Omega_{5\sigma}} g\left(\varphi^p(u-v)\right)_{x_n} dx.$$

By monotonicity (3.4), for $p \geq 2$

$$\tilde{\lambda}\fint_{\Omega_{5\sigma}} \varphi^p |Du - Dv|^p dx \leq J_1,$$

and for $p < 2$, using Young's inequality, (3.62) and (3.34), we have

$$\begin{aligned}
\tilde{\lambda}\fint_{\Omega_{5\sigma}} \varphi^p |Du - Dv|^p dx &= \tilde{\lambda}\fint_{\Omega_{5\sigma}} \left(|Du| + |Dv|\right)^{\frac{p(2-p)}{2}} \\
&\qquad \cdot \left[\left(|Du| + |Dv|\right)^{\frac{p(p-2)}{2}} \varphi^p |Du - Dv|^p\right] dx \\
&\leq \tilde{\lambda}\fint_{\Omega_{5\sigma}} \kappa\left(|Du| + c\right)^p + c(\kappa)\left(|Du| + |Dv|\right)^{p-2} \varphi^2 |Du - Dv|^2 dx \\
&\leq \tilde{\lambda}\kappa\fint_{\Omega_{5\sigma}} \left(|Du| + c\right)^p dx \\
&\qquad + c(\kappa)\fint_{\Omega_{5\sigma}} \varphi^p \langle \mathbf{a}(Du,x) - \mathbf{a}(Dv,x), Du - Dv\rangle \, dx \\
&\leq c_1\kappa + c_1(\kappa)J_1.
\end{aligned}$$

Put $\mathbf{a}(x,\xi) = (a^1(x,\xi), \cdots, a^n(x,\xi))$. Using the mean value theorem to

$a^i(x,\xi)$ in ξ and Young's inequality, we have

$$\begin{aligned} a^i(x,Du) - a^i(x,Dv) &= \int_0^1 \left\langle D_{\xi_i} a^i(x, sDu + (1-s)Dv), Du - Dv \right\rangle ds \\ &\le \Lambda \int_0^1 |s(Du-Dv) + Dv|^{p-2} ds\, |Du - Dv| \\ &\le c\left(|Du-Dv|^{p-1} + |Dv|^{p-2}|Du-Dv|\right) \\ &\le c\left(|Du-Dv|^{p-1} + |Dv|^{p-1}\right). \end{aligned}$$

Then applying Young's inequality with τ and Hölder inequality to J_2,

$$\begin{aligned} |J_2| &\le c \fint_{\Omega_{5\sigma}} \varphi^{p-1}|u-v|(|Du-Dv|^{p-1} + |Dv|^{p-1})|D\varphi| dx \\ &\le c \fint_{\Omega_{5\sigma}} \varphi^{p-1}|u-v||Du-Dv|^{p-1} + \varphi^{p-1}|u-v||Dv|^{p-1} dx \\ &\le c(\tau) \fint_{\Omega_{5\sigma}} |u-v|^p dx + c\tau \fint_{\Omega_{5\sigma}} |Du-Dv|^p \varphi^p dx + c\left(\fint_{\Omega_{5\sigma}} |u-v|^p dx\right)^{\frac{1}{p}}. \end{aligned}$$

Furthermore, Hölder inequality, Sobolev inequality and (3.58) imply for $1 < p < n$

$$\begin{aligned} \fint_{\Omega_{5\sigma}} |u-v|^p dx &= \frac{1}{|\Omega_{5\sigma}|}\left(\int_{\Omega_{5\sigma}\cap F_6^*} |u-v|^p dx + \int_{\Omega_{5\sigma}\setminus F_6^*} |u|^p dx\right) \\ &\le c\eta^p + c\left(\int_{\Omega_{5\sigma}\setminus F_6^*} |u|^{\frac{np}{n-p}} dx\right)^{\frac{n-p}{n}} |\Omega_{5\sigma} \setminus F_6^*|^{\frac{p}{n}} \\ &\le c\eta^p + c\delta^{\frac{p}{n}} \int_{\Omega_{5\sigma}} |Du|^p dx \\ &\le c\left(\eta^p + \delta^{\frac{p}{n}}\right) \end{aligned}$$

and for $p \geq n$

$$\begin{aligned}
\fint_{\Omega_{5\sigma}} |u-v|^p dx &= \frac{1}{|\Omega_{5\sigma}|}\left(\int_{\Omega_{5\sigma}\cap F_6^*} |u-v|^p dx + \int_{\Omega_{5\sigma}\setminus F_6^*} |u|^p dx\right) \\
&\leq c\eta^p + c\left(\int_{\Omega_{5\sigma}\setminus F_6^*} |u|^{2p} dx\right)^{\frac{1}{2}} |\Omega_{5\sigma}\setminus F_6^*|^{\frac{1}{2}} \\
&\leq c\eta^p + c\delta^{\frac{1}{2}} \int_{\Omega_{5\sigma}} |Du|^p dx \\
&\leq c\left(\eta^p + \delta^{\frac{1}{2}}\right).
\end{aligned}$$

So we get

$$|J_2| \leq c\tau \int_{\Omega_{5\sigma}} \varphi^p |Du - Dv|^p dx + c(\tau)\eta^p + c\eta + c\delta^{\tilde{p}},$$

where

$$\tilde{p} = \tilde{p}(p,n) := \begin{cases} \frac{p}{n} & 1 < p < n, \\ \frac{1}{2p} & p \geq n. \end{cases}$$

By Definition 3.1.2 and (3.62), we can see that for a.e. $x \in \Omega_5\sigma$

$$\begin{aligned}
\left|\bar{\mathbf{a}}_{\Omega_6}\left(Dv(x)\right) - \bar{\mathbf{a}}_{F_6^*}\left(Dv(x)\right)\right| &\leq \fint_{F_6^*} \left|\mathbf{a}\left(y, Dv(x)\right) - \bar{\mathbf{a}}_{\Omega_6}\left(Dv(x)\right)\right| dy \\
&\leq \fint_{F_6^*} \beta(\mathbf{a}, \Omega_6)(y)\left(1 + \|Dv\|^{p-1}_{L^\infty(\Omega_5\sigma)}\right) dy \\
&\leq c\frac{|\Omega_6|}{|F_6^*|}\left(\fint_{\Omega_6} \beta(\mathbf{a}, \Omega_6)(y)^p dy\right)^{\frac{1}{p}} \\
&\leq c\left(1 + \frac{192\delta}{\sigma^3}\right)^n \delta = c\delta.
\end{aligned}$$

Using this estimate, (3.62), (3.34) and (3.36), we have

$$
\begin{aligned}
|J_3| &\leq \fint_{\Omega_{5\sigma}} \varphi^p \left|\mathbf{a}(x, Dv) - \bar{\mathbf{a}}_{F_6^*}(Dv)\right| |Du - Dv| dx \\
&\leq \fint_{\Omega_{5\sigma}} \varphi^p \left(|\mathbf{a}(x, Dv) - \bar{\mathbf{a}}_{\Omega_6}(Dv)| + \left|\bar{\mathbf{a}}_{\Omega_6}(Dv) - \bar{\mathbf{a}}_{F_6^*}(Dv)\right|\right) |Du - Dv| dx \\
&\leq \fint_{\Omega_{5\sigma}} \varphi^p \beta(\mathbf{a}, \Omega_6) \left(1 + \|Dv\|^{p-1}_{L^\infty(\Omega_{5\sigma})}\right) |Du - Dv| dx + c\delta \fint_{\Omega_{5\sigma}} \varphi^p |Du - Dv| dx, \\
&\leq c \underbrace{\fint_{\Omega_{5\sigma}} \varphi^p \beta(\mathbf{a}, \Omega_6) |Du - Dv| dx}_{J_3^1} + c\delta.
\end{aligned}
$$

In the case that $p \geq 2$,

$$
\begin{aligned}
|J_3^1| &\leq c\tau \fint_{\Omega_{5\sigma}} \varphi^p |Du - Dv|^p dx + c(\tau) \fint_{\Omega_{5\sigma}} \beta(\mathbf{a}, \Omega_6)^{\frac{p}{p-1}} dx \\
&\leq c\tau \fint_{\Omega_{5\sigma}} \varphi^p |Du - Dv|^p dx + c(\tau) \left(\fint_{\Omega_{5\sigma}} \beta(\mathbf{a}, \Omega_6)^p dx\right)^{\frac{1}{p-1}} \\
&\leq c\tau \fint_{\Omega_{5\sigma}} \varphi^p |Du - Dv|^p dx + c(\tau) \delta^{\frac{p}{p-1}}.
\end{aligned}
$$

In the case that $1 < p < 2$,

$$
\begin{aligned}
|J_3^1| &\leq c \fint_{\Omega_{5\sigma}} \varphi^p \beta(\mathbf{a}, \Omega_6) |Du - Dv| dx \\
&\leq c\tau \fint_{\Omega_{5\sigma}} \varphi^p |Du - Dv|^p dx + c(\tau) \fint_{\Omega_{5\sigma}} \beta(\mathbf{a}, \Omega_6)^{\frac{p}{p-1}} dx \\
&= c\tau \fint_{\Omega_{5\sigma}} \varphi^p |Du - Dv|^p dx + c(\tau) \fint_{\Omega_{5\sigma}} \beta(\mathbf{a}, \Omega_6)^{\frac{1}{p-1}} \beta(\mathbf{a}, \Omega_6) dx \\
&\leq c\tau \fint_{\Omega_{5\sigma}} \varphi^p |Du - Dv|^p dx + c(\tau) \left(\fint_{\Omega_{5\sigma}} \beta(\mathbf{a}, \Omega_6)^p\right)^{\frac{1}{p}} dx \\
&\leq c\tau \fint_{\Omega_{5\sigma}} \varphi^p |Du - Dv|^p dx + c(\tau)\delta.
\end{aligned}
$$

So

$$|J_3| \le c\tau \fint_{\Omega_{5\sigma}} \varphi^p |D(u-v)|^p dx + c(\tau)\delta + c\delta.$$

Using Young's inequality, (3.35) and (3.58), we have

$$\begin{aligned}
|J_4| &\le c \fint_{\Omega_{5\sigma}} \varphi^{p-1} |u-v| \left(\beta(\mathbf{a}, \Omega_6) + \delta\right) dx \\
&\le c \fint_{\Omega_{5\sigma}} |u-v|^p + \beta(\mathbf{a}, \Omega_6)^{\frac{p}{p-1}} + \delta^{\frac{p}{p-1}} dx \\
&\le c \fint_{\Omega_{5\sigma}} |u-v|^p dx + \delta + \delta^{\frac{p}{p-1}} \\
&\le c(\eta^p + \delta),
\end{aligned}$$

here the estimate of $\left|\mathbf{a}(x, Dv) - \bar{\mathbf{a}}_{F_6^*}(Dv)\right|$ is done already in estimating J_3.

By Young's inequality with τ and (3.35), we have

$$\begin{aligned}
|J_5| &\le \fint_{\Omega_{5\sigma}} \varphi^p |\mathbf{f}|^{p-1} |Du - Dv| dx \\
&\le c\tau \fint_{\Omega_{5\sigma}} \varphi^p |Du - Dv|^p dx + c(\tau)\delta^p.
\end{aligned}$$

Applying Young's inequality, (3.35) and (3.58), we have

$$\begin{aligned}
|J_6| &\le \fint_{\Omega_{5\sigma}} p\varphi^{p-1} |u-v| |\mathbf{f}|^{p-1} |D\varphi| dx \\
&\le c(\delta^p + \eta^p).
\end{aligned}$$

Recalling (3.60) and using (3.61), (3.62), Hölder inequality, Poincaré-

Sobolev inequality and (3.34), we have

$$
\begin{aligned}
|J_7| &\le \fint_{\Omega_{5\sigma}} \left| g \left(p\varphi^{p-1}\varphi_{x_n}(u-v) + \varphi u_{x_n} - v_{x_n} \right) \right| dx \\
&\le \frac{|\Omega_{4\sigma} \setminus F_6^*|}{|\Omega_{5\sigma}|} \fint_{\Omega_{4\sigma}\setminus F_6^*} |g| \left| p\varphi^{p-1}\varphi_{x_n} u + \varphi u_{x_n} \right| dx \\
&\le c\delta \fint_{\Omega_{4\sigma}\setminus F_6^*} |Dv\left(x', \gamma(x')\right)|^{p-1}(|u| + |Du|) dx \\
&\le c\delta \fint_{\Omega_{4\sigma}\setminus F_6^*} |u| + |Du| dx \\
&\le c\delta \left(\fint_{\Omega_{4\sigma}\setminus F_6^*} |u|^p + |Du|^p dx \right)^{\frac{1}{p}} \\
&\le c\delta \left(\frac{|\Omega_6|}{|\Omega_{4\sigma} \setminus F_6^*|} \fint_{\Omega_6} |Du|^p dx \right)^{\frac{1}{p}} \\
&\le c\delta^{\frac{p-1}{p}}.
\end{aligned}
$$

Now we gather above estimates for J_1 to J_7 to find that for $p \ge 2$

$$
\begin{aligned}
\tilde{\gamma} \fint_{\Omega_{5\sigma}} \varphi^p |Du - Dv|^p dx \le |J_1| \le |J_2| + \cdots + |J_7| \\
\le c\tau \fint_{\Omega_{5\sigma}} \varphi^p |Du - Dv|^p dx + c(\tau)(\eta^p + \delta^p + \delta) \\
+ c\left(\eta + \eta^p + \delta + \delta^{\tilde{p}} + \delta^{\frac{p-1}{p}} \right).
\end{aligned}
$$

Now we recall (3.63) and take τ so small, to discover that

$$
\fint_{\Omega_{3\sigma}} |Du - Dv|^p dx \le c\left(\eta + \eta^p + \delta + \delta^p + \delta^{\tilde{p}} + \delta^{\frac{p-1}{p}} \right) \le \epsilon^p,
$$

by selecting η and δ so that the last inequality holds.

And, gathering above estimates for J_1 to J_7 for $1 < p < 2$, we find

$$\begin{aligned}\tilde{\gamma} \fint_{\Omega_{5\sigma}} \varphi^p |Du - Dv|^p dx &\le c_1\kappa + c_1(\kappa)|J_1| \\ &\le c_1\kappa + c_1(\kappa)\left(|J_2| + \cdots + |J_7|\right) \\ &\le c_1(\kappa)\tau \fint_{\Omega_{5\sigma}} \varphi^p |Du - Dv|^p dx \\ &\quad + c_1\kappa + c_1(\kappa)c(\tau)(\eta^p + \delta^p + \delta) + c_1(\kappa)\left(\eta + \eta^p + \delta + \delta^{\tilde{p}} + \delta^{\frac{p-1}{p}}\right).\end{aligned}$$

We take first κ to be $c_1\kappa = \frac{\tilde{\gamma}\epsilon^p}{4}$ in order to have

$$\begin{aligned}\tilde{\gamma} \fint_{\Omega_{5\sigma}} \varphi^p |Du - Dv|^p dx &\le c\tau \fint_{\Omega_{5\sigma}} \varphi^p |Du - Dv|^p dx \\ &\quad + \frac{\tilde{\gamma}\epsilon^p}{4} + c(\tau)(\eta^p + \delta^p + \delta) + c\left(\eta + \eta^p + \delta + \delta^{\tilde{p}} + \delta^{\frac{p-1}{p}}\right).\end{aligned}$$

We take τ so small that we find

$$\fint_{\Omega_{3\sigma}} |Du - Dv|^p dx \le \frac{\epsilon^p}{2} + c\left(\eta + \eta^p + \delta + \delta^p + \delta^{\tilde{p}} + \delta^{\frac{p-1}{p}}\right) \le \epsilon^p,$$

by selecting η and δ so that the last inequality holds. Then we obtain

$$\fint_{\Omega_{3\sigma}} |D(u - v)|^p dx \le \epsilon^p.$$

□

The following lemma is the global version of Lemma 3.4.1. We use the maximal function to show how the upper level sets of $|Du|$ decay.

Lemma 3.4.5. *Assume that $u \in W_0^{1,p}(\Omega)$ is the weak solution of (3.1). Then there is a constant $N_0 = N_0(\lambda, \Lambda, n, p) > 1$ so that for any $0 < \epsilon < 1$ fixed, one can find a small constant $\delta = \delta(\epsilon) > 0$ such that if $\mathbf{a}$ is $(\delta, \frac{48}{\sigma})$-vanishing, Ω is $(\delta, \sigma, \frac{48}{\sigma})$-quasiconvex, and $B_r(y)$, $0 < r \le 1$, $y \in \Omega$, satisfies*

$$|\{x \in \Omega : \mathcal{M}\left(|Du|^p\right)(x) > N_0^p\} \cap B_r(y)| \ge \epsilon |B_r(y)|, \tag{3.64}$$

then we have

$$\Omega_r(y) \subset \{x \in \Omega : \mathcal{M}(|Du|^p)(x) > 1\} \cup \{x \in \Omega : \mathcal{M}(|\mathbf{f}|^p)(x) > \delta^p\}. \quad (3.65)$$

Proof. If $B_{\frac{9r}{\sigma}}(y) \subset \Omega$, we can apply the interior estimate in Lemma 3.4.1 to find a constant $N_1 > 0$ for which this lemma holds true.

Now we assume that there is a boundary point $y_0 \in B_{\frac{9r}{\sigma}}(y) \cap \partial\Omega$. For this case, we argue by contradiction. If $B_r(y)$ satisfies (3.64) and the conclusion (3.65) is false, then there exists $x^0 \in \Omega_r(y)$ such that

$$\fint_{\Omega_\rho(x^0)} |Du|^p dx \leq 1 \quad \text{and} \quad \fint_{\Omega_\rho(x^0)} |\mathbf{f}|^p dx \leq \delta^p \quad (3.66)$$

for all $\rho > 0$. Since $0 < \sigma < 1$, $y_0 \in B_{\frac{9r}{\sigma}}(y) \cap \partial\Omega$ and $x^0 \in \Omega_r(y)$, we have $x^0 \in B_{\frac{10r}{\sigma}}(y_0)$. Since we eventually need the approximation estimate (3.57) on $\Omega_{\frac{12r}{\sigma}}(y_0)$, we consider a weak solution on $\Omega_{\frac{24r}{\sigma}}(y_0)$ and observe that $\Omega_{\frac{32r}{\sigma}}(y_0) \subset \Omega_{\frac{48r}{\sigma}}(x^0)$. Then we obtain from (3.66) that

$$\begin{aligned} &\fint_{\Omega_{\frac{24r}{\sigma}}(y_0)} |Du|^p dx \leq \frac{\left|\Omega_{\frac{30r}{\sigma}}(x^0)\right|}{\left|\Omega_{\frac{24r}{\sigma}}(y_0)\right|} \fint_{\Omega_{\frac{30r}{\sigma}}(x^0)} |Du|^p dx \leq \left(\frac{5}{4\sigma}\right)^n \quad \text{and} \\ &\fint_{\Omega_{\frac{24r}{\sigma}}(y_0)} |\mathbf{f}|^p dx \leq \frac{\left|\Omega_{\frac{30r}{\sigma}}(x^0)\right|}{\left|\Omega_{\frac{24r}{\sigma}}(y_0)\right|} \fint_{\Omega_{\frac{30r}{\sigma}}(x^0)} |\mathbf{f}|^p dx \leq \left(\frac{5}{4\sigma}\right)^n \delta^p. \end{aligned} \quad (3.67)$$

The change of variables

$$x \to y_0 + 4rx$$

maps $\Omega_{24r}(y_0)$ into $\tilde{\Omega}_6$ for $\tilde{\Omega} = \left\{\frac{x-y_0}{4r} \,|\, x \in \Omega\right\}$. Denoting with $\tilde{x}$ the new variable, the rescaled function

$$\tilde{u}(\tilde{x}) := \left(\frac{3}{2\sigma}\right)^{\frac{n}{p}} \frac{1}{4r} u\left(\frac{\tilde{x} - y_0}{4r}\right)$$

is a weak solution of (3.15) in Ω_6 with

$$\tilde{\mathbf{a}}(\tilde{x}, \xi) = \left(\frac{3}{2\sigma}\right)^{\frac{n(p-1)}{p}} \mathbf{a}\left(\frac{\tilde{x} - y_0}{4r}, \left(\frac{2\sigma}{3}\right)^{\frac{n}{p}} \xi\right) \quad \text{and} \quad \tilde{\mathbf{f}}(\tilde{x}) = \left(\frac{3}{2\sigma}\right)^{\frac{n}{p}} \mathbf{f}\left(\frac{\tilde{x} - y_0}{4r}\right).$$

With this setting, we can check the hypotheses of Corollary 3.4.4, using Lemma 3.2.6 and (3.67), as follows.

1. From Lemma 3.2.6, if Ω is $(\delta, \sigma, \frac{32}{\sigma})$-quasiconvex, then $\tilde{\Omega}$ is $(\delta, \sigma, \frac{8}{r})$-quasiconvex. But then since $r \in (0, 1]$, $\tilde{\Omega}$ is $(\delta, \sigma, 8)$-quasiconvex. As a consequence, (3.17) holds for some convex sets $\tilde{F}_6$ and $\tilde{F}_6^*$ associated with $\tilde{\Omega}_6$.

2. By the same reasoning, $\tilde{\mathbf{a}}$ is $(\delta, 8\sigma)$-vanishing. Hence
$$\fint_{\tilde{\Omega}_6} |\beta(\tilde{\mathbf{a}}, \tilde{\Omega}_6)|^p \, d\tilde{x} \leq \delta^p.$$

3. Lastly, (3.67) implies that
$$\fint_{\tilde{\Omega}_6} |D\tilde{u}|^p d\tilde{x} \leq 1 \quad \text{and} \quad \fint_{\tilde{\Omega}_6} |\tilde{\mathbf{f}}|^p d\tilde{x} \leq \delta^p.$$

Therefore, by Corollary 3.4.4, we find that there exists a weak solution of (3.16), $\tilde{v}$ such that
$$\fint_{\tilde{F}_6^*} |D\tilde{v}|^p dx \leq 1 \quad \text{and} \quad \fint_{\tilde{\Omega}_{3\sigma}} |D(\tilde{u} - \tilde{v})|^p \, dz \leq c_* \epsilon^p, \tag{3.68}$$
where $\tilde{v}$ is extended from $\tilde{F}_6^*$ to $\tilde{\Omega}_6$ by the zero extension, and c_* is to be determined later. Changing variable back, we discover from (3.68) that
$$\fint_{\Omega_{12r}(y_0)} |D(u - v)|^p \, dx \leq \left(\frac{5}{4\sigma}\right)^n c_* \epsilon^p. \tag{3.69}$$

Then, in view of Lemma 3.4.2 and (3.68), we have a constant $N_3 > 0$ such that
$$\begin{aligned} \|Dv\|^p_{L^\infty(\Omega_{12r}(y_0))} &\leq \|Dv\|^p_{L^\infty\left(B_{\frac{8r}{\sigma}}(y_0) \cap F^*_{\frac{24r}{\sigma}}(y_0)\right)} = \|D\tilde{v}\|^p_{L^\infty(B_{2r} \cap \tilde{F}_6^*)} \\ &\leq c \left(\fint_{\tilde{F}_6^*} |D\tilde{v}|^p d\tilde{x} + 1 \right) \leq N_3^p, \end{aligned}$$

where relation between v and $\tilde{v}$ is $\tilde{v}(\tilde{x}) := \left(\frac{5}{4\sigma}\right)^{\frac{n}{p}} \frac{\sigma}{4r} v\left(\frac{\sigma(\tilde{x}-y_0)}{4r}\right)$. Let's denote $N_2^p = \max\left\{2^{p+1}N_3^p, \left(\frac{3}{\sigma}\right)^n\right\}$, and claim that

$$\{x \in \Omega_{12r} : \mathcal{M}\left(|Du|^p\right)(x) > N_2^p\} \subset \{x \in \Omega_{12r} : \mathcal{M}\left(|D(u-v)|^p\right)(x) > N_3^p\}. \tag{3.70}$$

To do this, take any point $x_1 \in \{x \in \Omega_{8r}(y_0) : \mathcal{M}\left(|D(u-v)|^p\right)(x) \leq N_3^p\}$. For $0 < \rho \leq 4r$, since $\Omega_\rho(x_1) \subset \Omega_{12r}(y_0)$, we have

$$\begin{aligned} \fint_{\Omega_\rho(x_1)} |Du|^p dx &\leq 2^p \fint_{\Omega_\rho(x_1)} \left(|D(u-v)|^p + |Dv|^p\right) dx \\ &\leq 2^{p+1} N_0^p \leq N_2^p. \end{aligned}$$

For $\rho > 4r$, since $\Omega_\rho(x_1) \subset \Omega_{4\rho}(x^0)$, we have

$$\fint_{\Omega_\rho(x_1)} |Du|^p dx \leq \left(\frac{3}{\sigma}\right)^n \fint_{\Omega_{4\rho}(x^0)} |Du|^p dx \leq \left(\frac{3}{\sigma}\right)^n \leq N_2^p.$$

Hence, our claim (3.70) holds true.

Consequently, by Lemma 3.2.3, (3.69) and (3.70), we obtain that for any $\epsilon > 0$,

$$\begin{aligned} &\left|\{x \in \Omega_r(y) : \mathcal{M}(|Du|^p)(x) > N_2^p\}\right| \\ &\quad \leq \left|\{x \in \Omega_{8r}(y_0) : \mathcal{M}\left(|D(u-v)|^p\right)(x) > N_3^p\}\right| \\ &\quad \leq \frac{c_1}{N_3^p} \int_{\Omega_{8r}(y_0)} |D(u-v)|^p dx \\ &\quad \leq \frac{c_1 8^n}{N_3^p} \left(\frac{3}{2\sigma}\right)^n c_* \epsilon^p |B_r| \\ &\quad < \epsilon |B_r|, \end{aligned}$$

by taking c_* satisfying

$$\frac{c_1 8^n}{N_3^p} \left(\frac{3}{2\sigma}\right)^n c_* = 1.$$

This is a contradiction to (3.64). Hence we found a constant N_2 for which this theorem holds true for the case $B_{\frac{4r}{\sigma}}(y) \nsubseteq \Omega$. Consequently, we can finish the proof by taking $N_0 = max\{N_1, N_2\}$. $\square$

Proof of Theorem 3.1.5.

Suppose that the weak solution $u \in W_0^{1,p}(\Omega)$ of (3.1) satisfies the assumptions in Theorem 3.1.5, and take δ and N_0 for a fixed $\epsilon > 0$ as in Lemma 3.4.5. To prove the main theorem, it suffices to show that

$$\|\mathbf{f}\|_{L^q(\Omega,\mathbb{R}^n)} \leq \delta \qquad \Rightarrow \qquad \|Du\|_{L^q(\Omega,\mathbb{R}^n)} \leq c. \tag{3.71}$$

In fact, (3.71) and the normalization with $\tilde{u} = \dfrac{\delta u}{\|\mathbf{f}\|_{L^q} + \mu}$ and $\tilde{\mathbf{f}} = \dfrac{\delta \mathbf{f}}{\|\mathbf{f}\|_{L^q} + \mu}$, $\mu > 0$, imply, after letting $\mu \to 0^+$, the desired result.

Under the assumption $\|\mathbf{f}\|_{L^q(\Omega,\mathbb{R}^n)} \leq \delta$, we write

$$C = \left\{x \in \Omega : \mathcal{M}\left(|Du|^p\right)(x) > N_0^p\right\},$$
$$D = \left\{x \in \Omega : \mathcal{M}\left(|Du|^p\right)(x) > 1\right\} \cup \left\{x \in \Omega : \mathcal{M}\left(|\mathbf{f}|^p\right)(x) > \delta^p\right\}.$$

Then, one can check the first hypothesis of Lemma 3.2.5 as follows;

$$\begin{aligned} |C| &\leq \frac{c}{N_0^p} \int_\Omega |Du|^p dx \\ &\leq \frac{c}{N_0^p} \int_\Omega |\mathbf{f}|^p dx \leq \frac{c}{N_0^p} \|\mathbf{f}\|^p_{L^q(\Omega,\mathbb{R}^n)} |\Omega|^{1-\frac{p}{q}} \\ &\leq \delta^p |\Omega|^{1-\frac{p}{q}} < \epsilon |B_1|, \end{aligned}$$

by choosing a small $\delta = \delta(\epsilon) > 0$, if necessary, in order to get the last inequality. Here we have used Lemma 3.2.3, (3.6), Hölder inequality, and smallness of $\mathbf{f}$ in order. Meanwhile, the second hypothesis of Lemma 3.2.5 follows directly from Lemma 3.4.5. Therefore, by Lemma 3.2.5, we have that

$$\begin{aligned} \left|\left\{x \in \Omega : \mathcal{M}\left(|Du|^p\right)(x) > N_0^p\right\}\right| \leq{}& \epsilon_1 \left|\left\{x \in \Omega : \mathcal{M}\left(|Du|^p\right)(x) > 1\right\}\right| \\ &+ \epsilon_1 \left|\left\{x \in \Omega : \mathcal{M}\left(|\mathbf{f}|^p\right)(x) > \delta^p\right\}\right|, \end{aligned} \tag{3.72}$$

for $\epsilon_1 = \left(\frac{5}{\sigma}\right)^n \epsilon$. Using a simple iteration argument from (3.72), we further have that

$$\left|\left\{x \in \Omega : \mathcal{M}\left(|Du|^p\right)(x) > N_0^{kp}\right\}\right| \le \epsilon_1^k \left|\left\{x \in \Omega : \mathcal{M}\left(|Du|^p\right)(x) > 1\right\}\right| \tag{3.73}$$
$$+ \sum_{i=1}^{k} \epsilon_1^i \left|\left\{x \in \Omega : \mathcal{M}\left(|\mathbf{f}|^p\right)(x) > \delta^p N_0^{(k-i)p}\right\}\right|.$$

We use Lemma 3.2.1 and Lemma 3.2.3 to find that

$$\begin{aligned}
\|Du\|^q_{L^q(\Omega,\mathbb{R}^n)} &\le c_2 \|\mathcal{M}\left(|Du|^p\right)\|^{\frac{q}{p}}_{L^{\frac{q}{p}}(\Omega,\mathbb{R}^n)} \\
&\le c_1 c_2 \left(|\Omega| + \sum_{k=1}^{\infty} \left(N_0^p\right)^{k\frac{q}{p}} \left|\left\{x \in \Omega : \mathcal{M}\left(|Du|^p\right) > \left(N_0^p\right)^k\right\}\right|\right) \\
&\le c|\Omega| + c \underbrace{\sum_{k=1}^{\infty} \left(N_0^p\right)^{k\frac{q}{p}} \left|\left\{x \in \Omega : \mathcal{M}\left(|Du|^p\right) > \left(N_0^p\right)^k\right\}\right|}_{S_1}.
\end{aligned}$$

Hence, it is enough to show that S_1 is bounded by a universal constant. This can be done as follows. Using Lemma 3.2.1, smallness of $\mathbf{f}$ and (3.73), we obtain

$$
\begin{aligned}
S_1 &= \sum_{k=1}^{\infty} (N_0^p)^{k\frac{q}{p}} \left|\{x \in \Omega : \mathcal{M}\left(|Du|^p\right) > (N_0^p)^k\}\right| \\
&\leq \sum_{k=1}^{\infty} N_0^{qk} \Bigg(\sum_{i=1}^{k} \epsilon_1^i \left|\{x \in \Omega : \mathcal{M}(|\mathbf{f}|^p) > \delta^p N_0^{p(k-i)}\}\right| \\
&\qquad\qquad + \epsilon_1^k \left|\{x \in \Omega : \mathcal{M}\left(|Du|^p\right) > 1\}\right| \Bigg) \\
&= \sum_{i=1}^{\infty} (N_0^q \epsilon_1)^i \sum_{k=i}^{\infty} N_0^{q(k-i)} \left|\{x \in \Omega : \mathcal{M}\left(\left|\delta^{-1}\mathbf{f}\right|^p\right) > N_0^{p(k-i)}\}\right| \\
&\quad + \sum_{k=1}^{\infty} (N_0^q \epsilon_1)^k \left|\{x \in \Omega : \mathcal{M}\left(|Du|^p\right) > 1\}\right| \\
&\leq \sum_{i=1}^{\infty} (N_0^q \epsilon_1)^i \left(|\Omega| + \sum_{k=1}^{\infty} N_0^{qk} \left|\{x \in \Omega : \mathcal{M}(|\delta^{-1}\mathbf{f}|^p) > N_0^{pk}\}\right| \right) \\
&\quad + \sum_{k=1}^{\infty} (N_0^q \epsilon_1)^k \left|\{x \in \Omega : \mathcal{M}\left(|Du|^p\right) > 1\}\right| \\
&\leq \sum_{i=1}^{\infty} (N_0^q \epsilon_1)^i \left(2|\Omega| + c\|\delta^{-1}\mathbf{f}\|^q_{L^q(\Omega,\mathbb{R}^n)} \right) c \leq \sum_{i=1}^{\infty} (N_0^q \epsilon_1)^i \left(2|\Omega| + c\right).
\end{aligned}
$$

Thus, if we choose $\epsilon > 0$ so small that $N_0^q \epsilon_1 = N_0^q \left(\frac{5}{\sigma}\right)^n \epsilon < 1$, then $S_1 < c$. Therefore

$$\|Du\|_{L^q(\Omega)} \leq c,$$

for some universal constant c. This completes the proof of Theorem 3.1.5.

3.5 Gradient estimate in Orlicz spaces

In this last section, we will show that our gradient estimate is still valid in Orlicz spaces. The notion of Orlicz space extends the usual notion of L^p space with $p \geq 1$(e.g. see [65]). The function $\phi(s) = s^p$ used for the definition of L^p is replaced by a more general convex function $\phi(s)$, which is called to be a Young function.

Definition 3.5.1. A function $\phi : [0, \infty) \to [0, \infty]$ is a Young function if it is an increasing, convex function that satisfies

$$\phi(0) = 0,\ \phi(\infty) = \lim_{s\to\infty} \phi(s) = \infty,\ \lim_{s\to 0} \frac{\phi(s)}{s} = 0 \text{ and } \lim_{s\to\infty} \frac{\phi(s)}{s} = \infty. \quad (3.74)$$

Given a Young function ϕ and a bounded domain Ω, the Orlicz class $K^{\phi}(\Omega)$ is the set of all measurable functions $u : \Omega \to \mathbb{R}$ satisfying

$$\int_{\Omega} \phi(|u|)dx < \infty.$$

Note that the Orlicz class $K^{\phi}(\Omega)$ is in general a convex set but not a vector space. In fact the Orlicz space $L^{\phi}(\Omega)$ is the smallest vector space containing $K^{\phi}(\Omega)$. However, if ϕ satisfies the following Δ_2-condition, then the Orlicz class $K^{\phi}(\Omega)$ is always a vector space and coincides with the Orlicz space $L^{\phi}(\Omega)$ as a vector space. In addition, $L^{\phi}(\Omega)$ is a Banach space with the following Luxemburg norm.

Definition 3.5.2. Let ϕ be a Young function. Then

1. ϕ is said to obey the Δ_2-condition, written as $\phi \in \Delta_2$, if there exists constant $\mu > 1$ such that
$$\phi(2s) \geq \mu\phi(s) \text{ for all } s \geq 0.$$
2. ϕ is said to satisfy ∇_2-condition, written as $\phi \in \nabla_2$, if there exists constant $c > 1$ such that
$$\phi(s) \geq \frac{\phi(cs)}{2c} \text{ for all } s \geq 0.$$
3. For Young function $\phi \in \Delta_2$ the Luxemburg norm $\|\cdot\|_{L^{\phi}(\Omega)}$ is defined as
$$\|u\|_{L^{\phi}(\Omega)} = \inf\left\{ s > 0\ :\ \int_{\Omega} \phi\left(\frac{|u|}{s}\right) dx \geq 1 \right\}.$$

Here, we assume $\phi \in \Delta_2 \cap \nabla_2$, since Δ_2 and ∇_2 conditions are essential conditions, for the type of regularity under consideration, according to recent results [11, 18, 19, 73, 74, 75]. Now, we return to the gradient estimates in

Orlicz spaces for the problem (3.1). As theorem 3.1.5, we can expect the relation below,

$$|\mathbf{f}|^p \in L^\phi(\Omega) \Rightarrow |Du|^p \in L^\phi(\Omega), \forall \phi \in \Delta_2 \cap \nabla_2. \tag{3.75}$$

To prove this, we state some properties of Orlicz spaces in a couple of lemmas.

Lemma 3.5.3. *[53] Let ϕ be a Young function $\phi \in \Delta_2 \cap \nabla_2$ and U a bounded domain in $\mathbb{R}$. If $g \in L^\phi(U)$, then $\mathcal{M}g \in L^\phi(U)$ and*

$$\int_U \phi(|g|)dx \leq \int_U \phi(\mathcal{M}g)dx \leq c \int_U \phi(|g|)dx,$$

where the constant c is independent of g.

Lemma 3.5.4. *[53] Suppose that g is a nonnegative and measurable function in $\mathbb{R}^n$ and has compact support in a bounded set $U \in \mathbb{R}^n$. Let $\nu > 0$ and $\lambda > 1$ be constants. Then for any Young function $\phi \in \Delta_2 \cap \nabla_2$,*

$$g \in L^\phi(U) \Leftrightarrow S = \sum_{k=1}^{\infty} \phi(\lambda^k) \left|\left\{x \in U : g(x) > \nu\lambda^k\right\}\right| < \infty$$

and

$$\frac{1}{c} S \leq \int_U \phi(|g|)dx \leq c(|U| + S),$$

where $c = c(\nu, \lambda, \phi)$.

We are now ready to prove (3.75). To do this, we first recall the decay estimate (3.73) in the previous section and we calculate

$$\begin{aligned}
&\sum_{k=1}^{\infty} \phi(N_1^{pk}) \left|\left\{x \in \Omega : \mathcal{M}\left(|Du|^p\right)(x) > N_1^{pk}\right\}\right| \\
&\leq \sum_{k=1}^{\infty} \phi(N_1^{pk}) \epsilon_1^k \left|\left\{x \in \Omega : \mathcal{M}\left(|Du|^p\right)(x) > 1\right\}\right| \\
&+ \sum_{k=1}^{\infty} \phi(N_1^{pk}) \sum_{i=1}^{k} \epsilon_1^i \left|\left\{x \in \Omega : \mathcal{M}\left(|\mathbf{f}|^p\right)(x) > \delta^p N_1^{(k-i)p}\right\}\right| \\
&= S_1 + S_2.
\end{aligned}$$

Since $\phi \in \Delta_2$ and $N_1 > 1$, there exits a constant $\mu_1 > 1$, depending on N_1^p, such that $\phi(N_1^p) < \mu_1 \phi(1)$. After iterating k times, we get $\phi([N_1^p]^k) < \mu_1^k \phi(1)$.

And then
$$S_1 \leq \phi(1)|\Omega| \sum_{k=1}^{\infty} (\mu_1 \epsilon_1)^k.$$
Using Fubini's theorem and same argument as S_1, we get
$$S_2 \leq \phi(1) \sum_{i=1}^{\infty} (\mu_1 \epsilon_1)^i \left(\sum_{k=1}^{\infty} \phi\left(N_1^{p(k-i)}\right) \left| \left\{ x \in \Omega : \mathcal{M}\left(|\mathbf{f}|^p\right) > \delta^p N_1^{p(k-i)} \right\} \right| \right).$$
And then, by lemma 3.5.3 and 3.5.4, we further estimate
$$S_2 \leq \phi(1) \sum_{i=1}^{\infty} (\mu_1 \epsilon_1)^i \left(c \int_{\Omega} \phi\left(|\mathbf{f}|^p\right) dx \right).$$
Now, we take ϵ so small that $\mu_1 \epsilon_1 < 1$, to get
$$\sum_{k=1}^{\infty} \phi(N_1^{pk}) \left| \left\{ x \in \Omega : \mathcal{M}\left(|Du|^p\right)(x) > N_1^{pk} \right\} \right| \leq c.$$
Therefore we know, by Lemma 3.5.4,
$$\mathcal{M}\left(|Du|^p\right) \in L^{\phi}(\Omega).$$
Finally, by Lemma 3.5.3, we obtain
$$|Du|^p \in L^{\phi}(\Omega).$$

Chapter 4

Stokes system

4.1 Overview

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with nonsmooth boundary $\partial\Omega$. In this paper, we consider the following generalized Stokes problem with inhomogeneous data:

$$\begin{cases} \text{div } (A(x)\nabla u) - \nabla p &= \text{div } \mathbf{F} \quad \text{in} \quad \Omega \\ \text{div } u &= 0 \qquad \text{in} \quad \Omega \\ u &= 0 \qquad \text{on} \quad \partial\Omega, \end{cases} \tag{4.1}$$

where $\mathbf{F} = (\mathbf{F}^i_\alpha)^n_{i,\alpha=1}$ is a given matrix-valued function in $L^q_\omega(\Omega)^{n^2}$ which we will specify later, as is the tensor matrix-valued function $A = \left(A^{\alpha\beta}_{ij}\right)^n_{i,j,\alpha,\beta=1}$: $\mathbb{R}^n \to \mathbb{R}^{n^2\times n^2}$, satisfying uniform ellipticity and boundedness, namely; there exist positive constants ν and L such that

$$\nu|\xi|^2 \leq A(x)\xi : \xi, \quad |A(x)| \leq L \quad \forall \xi \in \mathbb{R}^{n^2}, \text{ a.e. } x \in \mathbb{R}^n. \tag{4.2}$$

Here $(\cdot : \cdot)$ denotes the standard inner product in $\mathbb{R}^{n^2}$ and the unknowns are the velocity $u = (u^1, \dots, u^n)$ and the pressure p.

The generalization of the classical steady Stokes system consists of general second order elliptic equations in divergence form instead of Laplace equations. This type of generalization can be found in [38, 67] and references given there. For this generalization, we allow the tensor matrix of coefficients A to be discontinuous, but we impose a small BMO (bounded mean oscillation) condition, as we now state.

Definition 4.1.1. We say that A is (δ, R)-vanishing if

$$\sup_{0<r\leq R}\sup_{x\in\mathbb{R}^n} \fint_{B_r(y)} \left|A(x)-\overline{A}_{B_r(y)}\right|^2 dx \leq \delta^2, \tag{4.3}$$

where $\overline{A}_{B_r(y)} = \fint_{B_r(y)} A(x)dx$ is the integral average of A over the open ball $B_r(y)$. We will clarify δ and R later after Definition 4.1.3.

Solvability and the regularity properties of solutions of the Stokes system form the fundamental part of fluid dynamics. In particular, there have been notable research activities on the boundary regularity regularity in the generalized Stokes system on the Lipschitz domain (see [43, 57]) and interior regularity of Stokes system (see [9, 26, 29, 30]). In the classical approach, which uses the representation formulas in terms of singular operators and commutators, one needs to overcome the obstacle coming from the non-graph domain, if one wants to deal with a nonsmooth domain beyond the Lipschitz category. In this situation, we cannot use directly the results obtained by using the representation formula, so we need other approach like a maximal function, as we will use here. The main goal of the present article is to develop a Calderón-Zygmund type theory for the steady Stokes system (4.1) in the setting of weighted Sobolev and Lebesgue spaces. This result will provide a new result in this literature, even for the unweighted case (in standard Sobolev and Lebesgue spaces).

We now introduce the definition of a weak solution pair of the problem (4.1).

Definition 4.1.2. Let $\mathbf{F} \in L^2(\Omega)^{n^2}$. Then $u \in W^{1,2}_{0,\sigma}(\Omega)^n$ is called a weak solution of the Stokes system (4.1), if

$$\int_\Omega A(x)\nabla u : \nabla\phi \, dx = \int_\Omega \mathbf{F} : \nabla\phi \, dx \tag{4.4}$$

holds for all $\phi \in W^{1,2}_{0,\sigma}(\Omega)^n = \{v \in W^{1,2}_0(\Omega)^n : \text{ div } v = 0\}$. If u is such a weak solution and $p \in L^2(\Omega)$ satisfies

$$\int_\Omega A(x)\nabla u : \nabla\phi - p \text{ div } \phi \, dx = \int_\Omega \mathbf{F} : \nabla\phi \, dx$$

for all $\phi \in W_0^{1,2}(\Omega)^n$, then (u, p) is called a weak solution pair to (4.1), and p is called an associated pressure of u.

In the next section we will return to the existence and uniqueness up to a constant of a weak solution pair to (4.1) over a bounded domain Ω with the following geometric regularity condition.

Definition 4.1.3. We say that $\Omega \subset \mathbb{R}^n$ is (δ, R)-Reifenberg flat if for every $x \in \partial\Omega$ and every $r \in (0, R]$, there exists an $(n-1)$ dimensional plane $L(x, r)$ passing through x such that

$$\frac{1}{r} D[\partial\Omega \cap B_r(x), L(x, r) \cap B_r(x)] \le \delta.$$

We can assume that R in both (4.3) and Definition4.1.3 equals to 1 by scaling the system, while δ is still invariant under such a scaling. Note that the concept of δ-Reifenberg flatness is a meaningful one for a small $\delta < \frac{1}{2^{n+1}}$ (see [70]). In this paper, we assume δ to be a small positive constant so that Ω, (δ, R)-Reifenberg flat domain, is also a non-tangentially accessible domain (see [52]). In particular, such domains are John domain (see [7]) and then Sobolev-Poincaré inequality holds on this domain (see [10]). For the properties of the Reifenberg flat domain, we refer to papers [42, 52, 60, 70].

We investigate this problem in weighted function spaces. More specifically, we consider Lebesgue spaces with respect to the measure ωdx, where ω is a weight in the Muckenhoupt class A_s with $1 < s < \infty$. This is the class of nonnegative and locally integrable weight function in $\mathbb{R}^n$, for which

$$[\omega]_s = \sup_{y \in \mathbb{R}^n} \sup_{r>0} \left(\fint_{B_r(y)} \omega(x)\, dx \right) \left(\fint_{B_r(y)} \omega(x)^{\frac{-1}{s-1}}\, dx \right)^{s-1} < \infty. \tag{4.5}$$

As in [36], typical examples of Muckenhoupt weights are

$$\omega(x) = |x|^{\alpha}, \quad -n < \alpha < n(q-1),$$

$$\omega(x) = \operatorname{dist}(x, \mathbf{M})^{\alpha}, \quad -(n-j) < \alpha < n(s-j)(s-1),$$

where $\mathbf{M}$ is a compact j-dimensional Lipschitzian submanifold. Therefore through choosing a particular weight function, the advanced theory can be used for a better control of the solution, such as in the neighborhood of a point or close to the boundary.

Given a weight $\omega \in A_s$, the weighted spaces is defined as

$$L^s_\omega(\Omega) = \left\{ f \in L^1_{loc}(\Omega) : \|f\|_{L^s_\omega(\Omega)} = \left(\int_\Omega |f(x)|^s \omega(x) dx \right)^{\frac{1}{s}} < \infty \right\},$$

and we let

$$\omega(E) = \int_E \omega(x)\, dx. \tag{4.6}$$

The main result of the paper is the following Calderón-Zygmund type estimate for a weak solution pair to (4.1).

Theorem 4.1.4 (Main result)**.** *Let $\omega \in A_{\frac{q}{2}}$ with $2 < q < \infty$. Assume $\mathbf{F} \in L^q_\omega(\Omega)^{n^2}$. Then there exists a small constant $\delta = \delta(n, q, \nu, L, \omega) > 0$ such that if $A(x)$ is (δ, R)-vanishing and $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is (δ, R)-Reifenberg flat domain, then a weak solution pair (u, p) satisfies*

$$\nabla u \in L^q_\omega(\Omega)^{n^2},\ p \in L^q_\omega(\Omega) \textit{ with the estimate}$$

$$\|\nabla u\|_{L^q_\omega(\Omega)^{n^2}} + \|p\|_{L^q_\omega(\Omega)} \leq c\|\mathbf{F}\|_{L^q_\omega(\Omega)^{n^2}}, \tag{4.7}$$

where the constant c depends only on $n, q, \nu, L, \omega, \Omega$.

Our result is an extension of the results in [17] to the context of Newtonian fluids and weighted spaces. In [17], the authors studied

$$\begin{aligned} \text{div } (A(x)\nabla u) &= \text{div } \mathbf{F} && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{4.8}$$

under similar conditions. In that article, Calderón-Zygmund type estimate for the weak solution to (4.8) was proved. This type estimate for an elliptic equation on the Reifenberg flat domain was first studied in [13] and then has been extended for system and parabolic problems.

The paper is organized as follows: In section 2, we introduce some notations and weighted Lebesgue space. Then we collect and prove the necessary statements needed for the main theorem. In section 3, we study global regularity of the gradient of weak solutions and an associated pressure to the Stokes system (4.1).

4.2 Preliminaries

4.2.1 Notations

1. $B_r(y) = \{x \in \mathbb{R}^n : |x - y| < r\}$ and $B_r = B_r(0)$.

2. $\Omega_r(y) = \Omega \cap B_r(y)$, $\Omega_r = \Omega_r(0)$, $B_r^+ = B_r \cap \{x \in \mathbb{R}^n : x_n > 0\}$ and $T_r = B_r \cap \{x \in \mathbb{R}^n : x_n = 0\}$.

3. $\partial\Omega$ is the boundary of the domain Ω, and $\partial_w\Omega_r(x) = \partial\Omega \cap B_r(x)$.

4. $|E|$ denotes the Lebesgue measure of the set $E \subset \mathbb{R}^n$.

5. $\overline{u}_E = \fint_E u(x)dx = \frac{1}{|E|}\int_E u(x)dx$ is the integral average of u over E.

6. For vector valued function $u : \Omega \to \mathbb{R}^n$, we write $u \in X^n$ if each component of u belongs to the function space X.

4.2.2 Weighted Lebesgue spaces and technical lemmas

An important property of the Muckenhoupt weight is the relation with the Lebesgue measure as in the following lemma.

Lemma 4.2.1. *[59] Let E be measurable subset of Ω and $\omega \in A_s$ for some $1 < s < \infty$. Then there exist positive constant μ and $\tau \in (0,1)$ independent of B, E, and ω such that*

$$\frac{1}{[\omega]_s}\left(\frac{|E|}{|B|}\right)^s \le \frac{\omega(E)}{\omega(B)} \le \mu\left(\frac{|E|}{|B|}\right)^\tau,$$

where B is a ball and E is a measurable subset of B.

Proof. From the [59, Lemma 3.3], A_s has strong doubling property,

$$\omega(B) \le [\omega]_s\left(\frac{|B|}{|E|}\right)^s \omega(E),$$

where B is a ball and E is a measurable subset of B. By [68, Proposition 9, V.5], we can see that there are ω_1 and ω_2 for each $\omega \in A_s$, $1 \leq s < \infty$, so that $\omega = \omega_1 \omega_2^{1-s}$. Then A_s has also reverse doubling property, using [69, Proposition 4.5, IX], as the following :

$$\omega(E) \leq \mu \left(\frac{|B|}{|E|} \right)^{\tau} \omega(B)$$

for some $\tau \in (0,1)$. Combining these two inequalities, the proof is completed. □

We introduce the following lemma, which is derived by the standard measure theory.

Lemma 4.2.2. *[59] Suppose that f is a nonnegative measurable function in a bounded domain Ω in $\mathbb{R}^n$ and $\omega \in A_s$, $1 < s < \infty$. Then*

$$f \in L^s_\omega(\Omega) \quad \textit{if and only if} \quad S = \sum_{k \geq 1} m^{ks} \omega \left(\{ x \in \Omega : f(x) > \theta m^k \} \right) < \infty$$

for some constants $\theta > 0$ and $m > 1$.

Moreover, we have

$$c^{-1} S \leq \|f\|^s_{L^s_\omega(\Omega)} \leq c(\omega(\Omega) + S),$$

where $c = c(\theta, m, s)$ is a positive constant.

In the following lemma, we observe basic properties of the Hardy-Littlewood maximal function.

Lemma 4.2.3. *[62, 68] Suppose $\omega \in A_s$ for some $s \in (1, \infty)$, then there exists a constant $C = C(n, s, [\omega]_s) > 0$ such that*

$$\frac{1}{C} \|f\|_{L^s_\omega(\mathbb{R}^n)} \leq \|\mathcal{M} f\|_{L^s_\omega(\mathbb{R}^n)} \leq C \|f\|_{L^s_\omega(\mathbb{R}^n)}.$$

In particular, if $s = 1$ and $\omega(x) \equiv 1$, then

$$\left| \{ x \in \mathbb{R}^n : (\mathcal{M})(x) > \nu \} \right| \leq \frac{C}{\nu} \int |f(x)| dx$$

for every $\nu > 0$, where $C = C(n)$.

For the global estimate on a $(\delta, 1)$-Reifenberg flat domain, we use weighted version of the Vitali covering lemma as following. We refer Lemma 3.8 in [59] for the proof.

Lemma 4.2.4. *[59] Let $\omega \in A_s$ for some $s \in (1, \infty)$ and let $\mathfrak{C}$ and $\mathfrak{D}$ are measurable sets, $\mathfrak{C} \subset \mathfrak{D} \subset \Omega$ where Ω, $(\delta, 1)$-Reifenberg flat with $0 < \delta < \frac{1}{8}$, Suppose further that there exists an $\epsilon > 0$ such that*

$$\omega(\mathfrak{C} \cap B_1(y)) < \epsilon\omega(B_1(y)) \text{ for all } y \in \Omega$$

and for all $y \in \Omega$ and for all $r \in (0, 1)$ it holds

$$B_r(y) \cap \Omega \subset \mathfrak{D} \text{ if } \omega(\mathfrak{C} \cap B_r(y)) \geq \epsilon\omega(B_r(y)).$$

Then,

$$\omega(\mathfrak{C}) \leq c^*\epsilon\omega(\mathfrak{D}),$$

where $c^* = c^*(n, s, [\omega]_s)$.

Another tool that makes our argument clean is scaling and normalization. Consider the following scaled and normalized setting: for $0 < \rho < 1$ and $\lambda > 1$,

$$\tilde{A}(x) = A(\rho x),\ \tilde{u}(x) = \frac{u(\rho x)}{\lambda\rho},\ \tilde{p}(x) = \frac{p(\rho x)}{\lambda},\ \tilde{\mathbf{F}}(x) = \frac{\mathbf{F}(\rho x)}{\lambda},\ \text{and } \tilde{\Omega} = \frac{1}{\rho}\Omega.$$

Then the following lemma holds.

Lemma 4.2.5.

1. *If (u, p) is a weak solution pair to (4.1), then $(\tilde{u}, \tilde{p})$ is a weak solution pair to*

$$\begin{cases} \operatorname{div}\,(\tilde{A}(x)D\tilde{u}) - \nabla\tilde{p} &= \operatorname{div}\,\tilde{\mathbf{F}} \quad \text{in} \quad \tilde{\Omega} \\ \operatorname{div}\,\tilde{u} &= 0 \quad \text{in} \quad \tilde{\Omega} \\ \tilde{u} &= 0 \quad \text{on} \quad \partial\tilde{\Omega}. \end{cases}$$

2. *If A satisfies the assumptions (4.2) and (4.3), then so does $\tilde{A}$ with the same constants ν and L.*

3. *If A is (δ, R)-vanishing in Ω, then $\tilde{A}$ is $\left(\delta, \frac{R}{\rho}\right)$-vanishing in $\tilde{\Omega}$.*

4. If Ω is (δ, R)-Reifenberg flat, then $\tilde{\Omega}$ is $\left(\delta, \frac{R}{\rho}\right)$-Reifenberg flat.

Proof. The proof follows from a direct computation. □

4.2.3 Existence and energy estimates of weak solution pairs

It is well known that if Ω is a bounded Lipschitz domain, then the following Stokes system

$$\begin{cases} -\Delta u + \nabla p &= \operatorname{div} \mathbf{F} \quad \text{in} \quad \Omega \\ \operatorname{div} u &= 0 \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

has a unique weak solution pair (u, p) with the energy estimate

$$\|\nabla u\|_{L^2(\Omega)^{n^2}} + \|p\|_{L^2(\Omega)} \leq c\|\mathbf{F}\|_{L^2(\Omega)^{n^2}},$$

for some positive constant $c = c(\Omega, n)$, see [39, 66].

Now we return to the generalized Stokes system (4.1). Here we claim that the problem (4.1) also has a unique weak solution pair with the standard estimate for our case that the underlying domain is (δ, R)-Reifenberg flat. It is well known that (δ, R)-Reifenberg flat domain is non-tangentially accessible for sufficiently small $\delta > 0$, it is a John domain as follows from [7, 52]. Roughly speaking, a domain is a John domain if it is possible to travel from one point to another without going too close to the boundary.

We need the following two lemmas regarding a John domain.

Lemma 4.2.6. *[1] Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded John domain. Given $f \in L^2(\Omega)$ such that $\int_\Omega f\,dx = 0$, there exists at least one $v \in W_0^{1,2}(\Omega)^n$ satisfying*

$$\operatorname{div} v = f \text{ in } \Omega,$$
$$\|\nabla v\|_{L^2(\Omega)^n} \leq C\|f\|_{L^2(\Omega)},$$

where $c = c(\Omega, n, q)$ but $c = c(n, q)$ if Ω is a ball.

Lemma 4.2.7. *[1, 39] Let $\Omega \in \mathbb{R}^n$, $n \geq 2$, be a John domain. Then any bounded linear functional $\mathscr{F}$ on $W_0^{1,2}(\Omega)^n$ identically vanishing on $W_{0,\sigma}^{1,2}(\Omega)^n$ is of the form $\mathscr{F}(v) = \int_\Omega p \operatorname{div} v \, dx$ for some uniquely determined $p \in \hat{L}^2(\Omega) := L^2(\Omega)/\mathbb{R}$.*

We now prove the existence and energy estimate of a weak solution pair to (4.1).

Lemma 4.2.8. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be an open bounded (δ, R)-Reifenberg flat domain with sufficiently small $\delta > 0$ and let $\mathbf{F} \in L^2(\Omega)^{n^2}$. Then there exists a unique solution pair $(u, p) \in W_{0,\sigma}^{1,2}(\Omega)^n \times L^2(\Omega)$ to (4.1) satisfying $\int_\Omega p \, dx = 0$ and the standard estimate*

$$\|\nabla u\|_{L^2(\Omega)^{n^2}} + \|p\|_{L^2(\Omega)} \leq C\|\mathbf{F}\|_{L^2(\Omega)^{n^2}}, \tag{4.9}$$

where $c = c(\Omega, n, \nu, L)$.

In addition, if $u \in W_{0,\sigma}^{1,q}(\Omega)^n$ and $\mathbf{F} \in L^q(\Omega)^{n^2}$ for $2 \leq q < \infty$, then

$$\|p\|_{L^q(\Omega)} \leq C \left(\|\mathbf{F}\|_{L^q(\Omega)^{n^2}} + \|\nabla u\|_{L^q(\Omega)^{n^2}} \right). \tag{4.10}$$

Proof. It is clear that $u \in W_{0,\sigma}^{1,2}(\Omega)$ is uniquely determined by applying Lax-Milgram theorem to (4.4). Using $u \in W_{0,\sigma}^{1,2}(\Omega)$ as the test function to (4.4), we have

$$\nu\|\nabla u\|_{L^2(\Omega)^{n^2}}^2 \leq \int_\Omega A(x)\nabla u : \nabla u dx = \int_\Omega \mathbf{F} : \nabla u dx \leq \|\mathbf{F}\|_{L^2(\Omega)^{n^2}} \|\nabla u\|_{L^2(\Omega)^{n^2}}$$

and so we have

$$\|\nabla u\|_{L^2(\Omega)^{n^2}} \leq \frac{1}{\nu}\|\mathbf{F}\|_{L^2(\Omega)^{n^2}}. \tag{4.11}$$

We next assume $u \in W^{1,q}(\Omega)^n$, $2 \leq q < \infty$ consider the functional

$$\mathscr{F}(v) \triangleq \int_\Omega A(x)\nabla u : \nabla v - \mathbf{F} : \nabla v \; dx, \qquad v \in W_0^{1,q'}(\Omega)^n, \tag{4.12}$$

where q' is Hölder conjugate. Note that if $v \in W_{0,\sigma}^{1,q}(\Omega)$, then $\mathscr{F}(v) = 0$. We also observe from (4.11) that

$$\begin{aligned} |\mathscr{F}(v)| &\leq c\|\nabla u\|_{L^q(\Omega)^{n^2}} \|\nabla v\|_{L^{q'}(\Omega)^{n^2}} + \|\mathbf{F}\|_{L^q(\Omega)^{n^2}} \|\nabla v\|_{L^{q'}(\Omega)^{n^2}} \\ &\leq c\|\mathbf{F}\|_{L^q(\Omega)^{n^2}} \|\nabla v\|_{L^{q'}(\Omega)^{n^2}} \end{aligned}$$

for some positive $c = c(n, \nu, L)$. Consequently, $\mathscr{F}$ is a bounded linear functional $\mathscr{F}$ on $W_0^{1,q'}(\Omega)^n$ identically vanishing on $W_{0,\sigma}^{1,2}(\Omega)^n$. Then since (δ, R)-Reifenbefg flat domain with small δ is a John domain, we apply Lemma 4.2.7 to discover that one can find a uniquely determined $p \in L^q(\Omega)$ with $\int_\Omega p\,dx = 0$ such that

$$\mathscr{F}(v) = \int_\Omega p \text{ div } v \ dx \tag{4.13}$$

for all $v \in W_0^{1,q'}(\Omega)$. Then we conclude that we have a unique solution pair $(u, p) \in W_{0,\sigma}^{1,2}(\Omega)^n \times L^2(\Omega)$ satisfying $\int_\Omega p\,dx = 0$, if $\mathbf{F} \in L^2(\Omega)^{n^2}$.

To prove (4.10), we consider the problem

$$\begin{aligned} &\text{div } v = |p|^{q-2}p - \fint_\Omega |p|^{q-2}p\,dx := g \\ &v \in W_0^{1,q'}(\Omega) \\ &\|v\|_{W^{1,q'}(\Omega)} \le C\|p\|_q^{q-1}. \end{aligned} \tag{4.14}$$

Since $\int_\Omega g\,dx = 0$, $g \in L^{q'}(\Omega)$, $\|g\|_{q'} \le c\|p\|_q^{q-1}$, from Lemma 4.2.6 we deduce the existence of v solving (4.14). If we replace such a v into (4.13) and use the assumption, $\int_\Omega p\,dx = 0$, together with the Hölder inequality, we see

$$\begin{aligned} \|p\|_q^q &= \int_\Omega |p|^q\,dx = \int_\Omega p\left(|p|^{q-2}p - \fint_\Omega |p|^{q-2}p\,dy\right)dx \\ &= \int_\Omega p \text{ div } v\,dx = \mathscr{F}(v) \\ &= \int_\Omega A(x)\nabla u : \nabla v + \mathbf{F} : \nabla v\,dx \\ &\le C\|\nabla u\|_q\|\nabla v\|_{q'} + \|\mathbf{F}\|_q\|\nabla v\|_{q'} \\ &\le C\left(\|\nabla u\|_q + \|\mathbf{F}\|_q\right)\|p\|_q^{q-1} \\ &\le C\left(\|\nabla u\|_q + \|\mathbf{F}\|_q\right)^q + \frac{1}{2}\|p\|_q^q \end{aligned}$$

This inequality yields (4.10). And (4.9) follows from (4.11) and (4.10). $\square$

A main point in this paper is that the nonhomogeneous term $\mathbf{F}$ belongs to a weighted Lebesgue space. More precisely,

$$|\mathbf{F}|^2 \in L^{\frac{q}{2}}_{\omega}(\Omega),\ \omega \in A_{\frac{q}{2}} \text{ for } q \in (2,\infty),$$

which means

$$\mathbf{F} \in L^q_{\omega}(\Omega)^{n^2},\ \omega \in A_{\frac{q}{2}} \subset A_q \text{ for } q \in (2,\infty).$$

Using Hölder inequality and (4.5), we compute

$$\begin{aligned}
\|\mathbf{F}\|^2_{L^2(\Omega)^{n^2}} &= \int_\Omega |\mathbf{F}|^2 \omega^{\frac{2}{q}} \omega^{-\frac{2}{q}} dx \\
&\leq \left(\int_\Omega \left(|\mathbf{F}|^2\right)^{\frac{q}{2}} \omega\, dx \right)^{\frac{2}{q}} \left(\int_\Omega \omega^{\frac{-2}{q-2}} dx \right)^{\frac{q-2}{q}} \\
&= \left\| |\mathbf{F}|^2 \right\|_{L^{\frac{q}{2}}_{\omega}(\Omega)^{n^2}} |\Omega|^{\frac{q-2}{q}} \left(\frac{1}{|\Omega|} \int_\Omega \omega^{\frac{-2}{q-2}} dx \right)^{\frac{q-2}{q}} \\
&\leq \left\| |\mathbf{F}|^2 \right\|_{L^{\frac{q}{2}}_{\omega}(\Omega)^{n^2}} |\Omega|^{-\frac{2}{q}} \omega(\Omega) [\omega]^{\frac{2}{q}}_{\frac{q}{2}},
\end{aligned}$$

which implies $\mathbf{F} \in L^2(\Omega)^{n^2}$. This guarantees the existence of a unique weak solution pair (u, p) to (4.1).

4.3 Gradient estimates in L^q_{ω}

Throughout this section we write c to mean any universal constant that can be explicitly computed in terms of known quantities such as ν, L, n, q, ω and the structure of Ω. Thus the exact value may vary from line to line. If necessary, we specify it by $c_1,\ c_2, \cdots$.

We first make interior comparison estimates. For doing this, we consider

$$\text{div } (A\nabla u) - \nabla p = \text{div } \mathbf{F},\ \text{div } u = 0 \text{ in } \Omega \supset B_6. \tag{4.15}$$

Suppose that

$$⨍_{B_5} |\nabla u|^2 + |p|^2 dx \leq 1. \tag{4.16}$$

As usual, a weak solution to (4.15) is a function $u \in W^{1,2}_\sigma(\Omega)^n$ such that

$$\int_\Omega A\nabla u : \nabla\phi\, dx = \int_\Omega \mathbf{F} : \nabla\phi\, dx$$

for all $\phi \in W^{1,2}_{0,\sigma}(\Omega)^n$, and (u,p) is a weak solution pair if and only if $u \in W^{1,2}_\sigma(\Omega)^n$ and $p \in L^2_{loc}(\Omega)$ satisfy

$$\int_\Omega A\nabla u : \nabla\phi - p\ \mathrm{div}\ \phi\, dx = \int_\Omega \mathbf{F} : \nabla\phi\, dx \tag{4.17}$$

for all $\phi \in W^{1,2}_0(\Omega)^n$.

We want to find local estimates of a weak solution pair to (4.17) in comparison with the homogeneous problem

$$\begin{cases} \mathrm{div}\ A\nabla h - \nabla p_h &= 0 \quad \text{in} \quad B_4 \\ \mathrm{div}\ h &= 0 \quad \text{in} \quad B_4 \\ h &= u \quad \text{on} \quad \partial B_4, \end{cases} \tag{4.18}$$

and the limiting problem

$$\begin{cases} \mathrm{div}\ \bar{A}_{B_4}\nabla v - \nabla p_v &= 0 \quad \text{in} \quad B_3 \\ \mathrm{div}\ v &= 0 \quad \text{in} \quad B_3 \\ v &= h \quad \text{on} \quad \partial B_3. \end{cases} \tag{4.19}$$

Taking the test function $h-u$ for (4.18) and $v-w$ for (4.19), respectively, and using (4.2) and (4.16), we have

$$⨍_{B_4} |\nabla h|^2 dx \le c ⨍_{B_4} |\nabla u|^2 dx \le c \text{ and } ⨍_{B_3} |\nabla v|^2 dx \le c ⨍_{B_3} |\nabla h|^2 dx \le c. \tag{4.20}$$

In what follows we need the following regularity results for v and h.

Lemma 4.3.1. *Let $h \in W^{1,2}_\sigma(B_4)^n$ be the weak solution to (4.18) satisfying (4.16). Then there exists an exponent $r_1 = r_1(\nu, L, n) > 2$ such that $\|\nabla h\|_{L^r(B_3)} \le c$.*

Proof. By Theorem 2.2 in [43], there is an exponent $r_1 = r_1(\nu, L, n) > 2$ such that

$$\int_{B_3} |\nabla h|^{r_1} dx \le \left(\int_{B_4} |\nabla h|^2 dx\right)^{\frac{r_1}{2}}.$$

But by (4.20) and (4.16), we have

$$\int_{B_4} |\nabla h|^2 dx \leq c \int_{B_4} |\nabla u|^2 dx \leq c.$$

The conclusion now follows immediately. $\square$

Lemma 4.3.2. *Let (v, p_v) be a weak solution pair to Stokes system (4.19) in B_3. Then there holds*

$$\|\nabla v\|_{L^\infty(B_2)^{n^2}} + \|p_v\|_{L^\infty(B_2)} \leq c.$$

Proof. According to a known regularity for the limiting problem (4.19), see [43], we have

$$\|\nabla v\|_{L^\infty(B_2)^{n^2}} + \|p_v\|_{L^\infty(B_2)} \leq c\|\nabla v\|^2_{L^2(B_3)^n}.$$

Then the conclusion follows from (4.20). $\square$

Lemma 4.3.3. *For any $0 < \epsilon < 1$, there exists a small $\delta = \delta(\epsilon) > 0$ such that if*

$$\fint_{B_4} |A - \bar{A}_{B_4}|^2 + |\mathbf{F}|^2 dx \leq \delta^2 \tag{4.21}$$

for any weak solution pair (u, p) to (4.4) with (4.16), then one can find a weak solution pair (v, p_v) to (4.19) in B_3 such that

$$\fint_{B_3} |\nabla(u - v)|^2 + |p - p_v|^2 dx \leq \epsilon^2.$$

Proof. Let (h, p_h) be a weak solution pair to (4.18). Then $(u - h, p - p_h) \in W^{1,2}_{0,\sigma}(B_4) \times L^2(B_4)$ is a weak solution pair to

$$\begin{cases} \text{div } A\nabla(u-h) - \nabla(p - p_h) &= \text{div } \mathbf{F} \quad \text{in} \quad B_4 \\ \text{div } (u-h) &= 0 \quad \text{in} \quad B_4 \\ u - h &= 0 \quad \text{on} \quad \partial B_4. \end{cases}$$

Using the Lemma 4.2.8 and (4.21), it follows that

$$\fint_{B_4} |\nabla(u - h)|^2 + |p - p_h|^2 dx \leq c\delta^2. \tag{4.22}$$

For a weak solution pair (v, v_p) to (4.19), $(h-v, p_h-p_v) \in W^{1,2}_{0,\sigma}(B_3)^n \times L^2(B_3)$ is a weak solution pair to

$$\begin{cases} \text{div } \bar{A}_{B_4}\nabla(h-v) - \nabla(p_h - p_v) &= -\text{div } \left((A - \bar{A}_{B_4})\nabla h\right) \quad \text{in} \quad B_3 \\ \text{div } (h-v) &= 0 \quad \text{in} \quad B_3 \\ h - v &= 0 \quad \text{on} \quad \partial B_3. \end{cases}$$

Then by Lemma 4.2.8, Hölder inequality and the boundedness of $A(x)$, we estimate

$$\begin{aligned} &\fint_{B_3} |\nabla(h-v)|^2 + |p_h - p_v|^2 dx \\ &\qquad \le c \fint_{B_3} \left|A - \bar{A}_{B_4}\right|^2 |\nabla h|^2 dx \\ &\qquad \le c \left(\fint_{B_3} \left|A - \bar{A}_{B_4}\right|^{\frac{2r_1}{r_1-2}} dx\right)^{\frac{r_1-2}{r_1}} \left(\fint_{B_3} |\nabla h|^{r_1} dx\right)^{\frac{2}{r_1}} \\ &\qquad \le c \left(\fint_{B_3} \left|A - \bar{A}_{B_4}\right|^2 \left|A - \bar{A}_{B_4}\right|^{\frac{4}{r_1-2}} dx\right)^{\frac{r_1-2}{r_1}} \\ &\qquad \le c \left(\fint_{B_4} \left|A - \bar{A}_{B_4}\right|^2 dx\right)^{\frac{r_1-2}{r_1}} \\ &\qquad \le c\delta^{2-\frac{4}{r_1}}. \end{aligned}$$

These estimates and (4.22) imply

$$\fint_{B_3} |\nabla(u-v)|^2 + |p - p_v|^2 dx \le c\left(\delta^2 + \delta^{2-\frac{4}{r_1}}\right) \le \epsilon^2,$$

by taking $\delta > 0$ so small that the last inequality holds. This finishes the proof. □

We next extend the interior comparison estimate obtained in Lemma 4.3.3 to find its boundary version. To do this, based on the definition of the (δ, R)-Reifenberg flatness, we are under the following geometric setting

$$B_6^+ \subset \Omega_6 \subset B_6 \cap \{x_n > -12\delta\}. \tag{4.23}$$

From now on we consider a localized problem, the homogeneous problem, the reference problem and a limiting problem as follows:

$$\begin{cases} \text{div } (A\nabla u) - \nabla p &= \text{div } \mathbf{F} \quad \text{in} \quad \Omega_6 \\ \text{div } u &= 0 \quad \text{in} \quad \Omega_6 \\ u &= 0 \quad \text{on} \quad \partial_w \Omega_6, \end{cases} \tag{4.24}$$

$$\begin{cases} \text{div } (A\nabla h) - \nabla p_h &= 0 \quad \text{in} \quad \Omega_5 \\ \text{div } h &= 0 \quad \text{in} \quad \Omega_5 \\ h &= u \quad \text{on} \quad \partial\Omega_5, \end{cases} \tag{4.25}$$

$$\begin{cases} \text{div } (\bar{A}_{B_6^+}\nabla w) - \nabla p_w &= 0 \quad \text{in} \quad \Omega_4 \\ \text{div } w &= 0 \quad \text{in} \quad \Omega_4 \\ w &= h \quad \text{on} \quad \partial\Omega_4, \end{cases} \tag{4.26}$$

and

$$\begin{cases} \text{div } (\bar{A}_{B_6^+}\nabla v) - \nabla p_v &= 0 \quad \text{in} \quad B_4^+ \\ \text{div } v &= 0 \quad \text{in} \quad B_4^+ \\ v &= 0 \quad \text{on} \quad T_4. \end{cases} \tag{4.27}$$

L^2-estimates for h and w are derived by selecting the test function $h - u$ for (4.25) and $w - h$ for (4.26), respectively, and computing in a typical way along with (4.2). We have

$$⨍_{\Omega_5} |\nabla h|^2 dx \leq c ⨍_{\Omega_5} |\nabla u|^2 dx \quad \text{and} \quad ⨍_{\Omega_4} |\nabla w|^2 dx \leq c ⨍_{\Omega_4} |\nabla h|^2 dx. \tag{4.28}$$

We further assume that

$$⨍_{\Omega_5} |\nabla u|^2 + |p|^2 dx \leq 1. \tag{4.29}$$

Then by (4.28), we discover that

$$⨍_{\Omega_5} |\nabla h|^2 dx \leq c \text{ and } ⨍_{\Omega_4} |\nabla w|^2 dx \leq c. \tag{4.30}$$

As in Lemma 4.3.1, the gradient of h, which is the weak solution to (4.25), has a higher integrability near the boundary. This is the following lemma.

Lemma 4.3.4. *Let $h \in W_\sigma^{1,2}(\Omega_5)^n$ be the weak solution to (4.25) with (4.23) satisfying $⨍_{\Omega_5} |\nabla h|^2 dx \leq 1$. Then there exists an exponent $r_2 =$*

$r_2(n,\Omega,\nu,L) > 2$ *such that* $\|\nabla h\|_{L^{r_2}(\Omega_4)} \le c$.

Proof. Since $h = u = 0$ on $\partial_w\Omega_5$, we may assume $h = 0$ in $B_5 \setminus \Omega_5$ by zero extension. Let $\eta \in C_0^\infty(B_5)$ be a standard cutoff function satisfying that $0 \le \eta \le 1$, $\eta = 1$ in B_4 and $|\nabla\eta| \le 2$, and let $\phi = \eta^2\bar{h} - \psi$, where $\bar{h} = h - (h)_{B_5}$ and ψ is defined by Lemma 4.2.6 as a solution to

$$\text{div } \psi = \text{div } (\eta^2\bar{h}) = \bar{h}\cdot\nabla(\eta^2) \text{ in } B_5.$$

Then ψ satisfies

$$\int_{B_5} |\nabla\psi|^2 dx \le c\int_{B_5} |\bar{h}\cdot\nabla(\eta^2)|^2 dx \le c\left(\int_{B_5} |\nabla h|^{2_*} dx\right)^{\frac{2}{2_*}}, \tag{4.31}$$

where we used Sobolev-Poincaré inequality for the last inequality and $2_* = \frac{2n}{n+2}$. Applying ϕ as a test function and doing standard computation with condition of $A(x)$ and Young's inequality, we obtain

$$\nu\int_{B_5} \eta^2|\nabla h|^2 dx \le \epsilon_1\int_{B_5} \eta^2|\nabla h|^2 dx + c_{\epsilon_1}\int_{B_5} |\nabla\eta|^2|\bar{h}|^2 dx + c\int_{B_5} |\nabla h||\nabla\psi| dx.$$

Employing Young's inequality and (4.31), we have

$$\begin{aligned}\int_{B_5} |\nabla h||\nabla\psi| dx &\le \epsilon_2\int_{B_5} |\nabla h|^2 dx + c_{\epsilon_2}\int_{B_5} |\nabla\psi|^2 dx \\ &\le \epsilon_2\int_{B_5} |\nabla h|^2 dx + c_{\epsilon_2}\left(\int_{B_5} |\nabla h|^{2_*} dx\right)^{\frac{2}{2_*}}.\end{aligned}$$

Taking ϵ_1 and ϵ_2 sufficiently small, we conclude that

$$\fint_{B_4} |\nabla h|^2 dx \le c\left(\fint_{B_5} |\nabla h|^{2_*} dx\right)^{\frac{2}{2_*}} + \epsilon\fint_{B_5} |\nabla h|^2 dx,$$

where $0 < \epsilon < 1$. Then we can apply Gehring lemma to arrive at the required inequality

$$\fint_{B_4} |\nabla h|^{r_2} dx \le c\left(\fint_{B_5} |\nabla h|^2 dx\right)^{\frac{r_2}{2}},$$

for some $r_2 = r_2(n,\nu,L)$. $\square$

We need the following better regularity for the limiting problem (4.27).

Lemma 4.3.5. *[43] Let (v, p_v) be a weak solution pair to the Stokes system (4.27) in B_4^+. Then we have*

$$\|\nabla v\|_{L^\infty(B_3^+)^{n^2}} \leq c\|\nabla v\|_{L^2(B_4^+)^{n^2}}$$

and

$$\|p_v\|_{L^\infty(B_3^+)} \leq c\left(\|\nabla v\|_{L^2(B_4^+)^{n^2}} + \|p_v\|_{L^2(B_4^+)}\right).$$

Lemma 4.3.6. *For any $0 < \epsilon < 1$, there exists a sufficiently small $\delta = \delta(\epsilon) > 0$ such that if (w, p_w) is a weak solution pair to (4.26) with (4.23) and the following normalization condition*

$$\fint_{\Omega_4} |\nabla w|^2 + |p_w|^2 dx \leq 1, \tag{4.32}$$

then there exists a weak solution pair (v, p_v) to (4.27) in B_4^+ with

$$\fint_{B_4^+} |\nabla v|^2 + |p_v|^2 dx \leq 1$$

such that

$$\fint_{B_4^+} |w - v|^2 dx \leq \epsilon^2.$$

Proof. We prove this lemma by contradiction. If not, then there exist $\epsilon_0 > 0$, $\{(w_k, p_{w_k})\}_{k=1}^\infty$, and $\{\Omega_4^k\}_{k=1}^\infty$ such that $(w_k, p_{w_k}) \in W_\sigma^{1,2}(\Omega_4^k)^n \times L^2(\Omega_4^k)$ is a weak solution pair to

$$\begin{cases} \operatorname{div}\,(\bar{A}_{B_6^+}\nabla w_k) - \nabla p_{w_k} &= 0 \quad \text{in} \quad \Omega_4^k \\ \operatorname{div}\, w_k &= 0 \quad \text{in} \quad \Omega_4^k \\ w_k &= 0 \quad \text{on} \quad \partial_w \Omega_4^k \end{cases} \tag{4.33}$$

with

$$B_6^+ \subset \Omega_6^k \subset B_6 \cap \left\{x_n > -\frac{12}{k}\right\} \text{ and } \fint_{\Omega_4^k} |\nabla w_k|^2 + |p_{w_k}|^2 dx \leq 1. \tag{4.34}$$

But it holds that

$$\fint_{B_4^+} |w_k - v|^2 dx > \epsilon_0^2 \tag{4.35}$$

for any weak solution v to (4.27) satisfying $\fint_{B_4^+} |\nabla v|^2 + |p_v|^2 dx \leq 1$.

Since we can say $(w_k, p_{w_k}) = 0$ in $B_4 \setminus \Omega_4$ by the zero extension from the fact $w = h = u = 0$ on $\partial_w \Omega_4$, (w_k, p_{w_k}) is uniformly bounded in $W_\sigma^{1,2}(B_4)^n \times L^2(B_4)$ in view of Poincaré inequality with (4.34). It implies that (w_k, p_{w_k}) is uniformly bounded in $W_\sigma^{1,2}(B_4^+)^n \times L^2(B_4^+)$. Thus there exists a subsequence, which we still denote by $\{(w_k, p_{w_k})\}$, and $(w_0, p_{w_0}) \in W_\sigma^{1,2}(B_4^+)^n \times L^2(B_4^+)$ such that

$$\begin{cases} w_k \rightharpoonup w_0 & \text{in} \quad W_\sigma^{1,2}(B_4^+)^n \\ w_k \to w_0 & \text{in} \quad L^2(B_4^+)^n \\ p_{w_k} \rightharpoonup p_{w_0} & \text{in} \quad L^2(B_4^+). \end{cases} \tag{4.36}$$

From (4.33), (4.34) and (4.35), it follows that

$$\begin{cases} \operatorname{div}\,(\bar{A}_{B_6^+} \nabla w_0) - \nabla p_{w_0} &= 0 \quad \text{in} \quad B_4^+ \\ \operatorname{div}\, w_0 &= 0 \quad \text{in} \quad B_4^+ \\ w_0 &= 0 \quad \text{on} \quad T_4. \end{cases}$$

Furthermore, it follows from (4.34) and (4.36) that

$$\fint_{B_4^+} |\nabla w_0|^2 + |p_0|^2 dx \leq \liminf_{k \to \infty} \fint_{B_4^+} |\nabla w_k|^2 + |p_k|^2 dx \leq 1.$$

Therefore, we reach a contradiction to (4.35) by (4.36). This completes the proof. $\square$

Lemma 4.3.7. *For any $0 < \epsilon < 1$, there exists a sufficiently small $\delta = \delta(\epsilon) > 0$ such that if (u, p) is a weak solution pair to (4.4) with (4.23) and the following normalization conditions*

$$\fint_{\Omega_5} |\nabla u|^2 + |p|^2 dx \leq 1 \text{ and } \fint_{\Omega_6} |\mathbf{F}|^2 + |A - \bar{A}_{\Omega_6}|^2 dx \leq \delta^2, \tag{4.37}$$

then there exists a weak solution pair (v, p_v) of (4.27) in B_3^+ with

$$\fint_{B_4^+} |\nabla v|^2 + |p_v|^2 dx \leq 1$$

such that

$$\fint_{\Omega_2} |\nabla u - \nabla V|^2 + |p - p_V|^2 dx \leq \epsilon^2.$$

where V is the zero extension of v from B_4^+ to B_4 and p_V is an associated pressure of V.

Proof. Let (h, p_h) and (w, p_w) be weak solution pairs to (4.25) and (4.26), respectively. Applying Lemma 4.2.8 to the system which is derived by subtracting (4.25) from (4.24), it follows that

$$\fint_{\Omega_5} |\nabla(u-h)|^2 + |p-p_h|^2 dx \le c \fint_{\Omega_5} |\mathbf{F}|^2 dx \le c\delta^2, \tag{4.38}$$

where the last inequality comes from (4.37).

By subtracting (4.26) from (4.25), we discover

$$\begin{cases} \operatorname{div} \bar{A}_{B_6^+} \nabla(h-w) - \nabla(p_h - p_w) &= -\operatorname{div} \left((A - \bar{A}_{B_6^+})\nabla h\right) & \text{in} & \Omega_4 \\ \operatorname{div} (h-w) &= 0 & \text{in} & \Omega_4 \\ h - w &= 0 & \text{on} & \partial\Omega_4. \end{cases}$$

Using Lemma 4.2.8, we compute

$$
\begin{aligned}
&\fint_{\Omega_4} |\nabla(h-w)|^2 + |p_h - p_w|^2 dx \\
&\qquad \le c \fint_{\Omega_4} |A - \bar{A}_{B_6^+}|^2 |\nabla h|^2 dx \\
&\qquad \le c \left(\fint_{\Omega_4} \left| A - \bar{A}_{B_6^+} \right|^{\frac{2r_2}{r_2-2}} dx \right)^{\frac{r_2-2}{r_2}} \left(\fint_{\Omega_4} |\nabla h|^{r_2} dx \right)^{\frac{2}{r_2}} \\
&\qquad \le c \left(\fint_{\Omega_4} \left| A - \bar{A}_{B_6^+} \right|^2 \left| A - \bar{A}_{B_6^+} \right|^{\frac{4}{r_2-2}} dx \right)^{\frac{r_2-2}{r_2}} \\
&\qquad \le c \left(\fint_{\Omega_6} \left| A - \bar{A}_{B_6^+} \right|^2 dx \right)^{\frac{r_2-2}{r_2}} \\
&\qquad \le c \left(\fint_{\Omega_6} \left| A - \bar{A}_{\Omega_6} \right|^2 dx + \left| \bar{A}_{\Omega_6} - \bar{A}_{B_6^+} \right|^2 \right)^{\frac{r_2-2}{r_2}} \\
&\qquad \le c \left(\delta^2 + \fint_{B_6^+} \left| A - \bar{A}_{\Omega_6} \right|^2 dx \right)^{\frac{r_2-2}{r_2}} \\
&\qquad \le c \left(\delta^2 + \frac{|\Omega_6|}{|B_6^+|} \fint_{\Omega_6} \left| A - \bar{A}_{\Omega_6} \right|^2 dx \right)^{\frac{r_2-2}{r_2}} \\
&\qquad \le c\big(\delta^2 + \delta^2(1+\delta)\big)^{\frac{r_2-2}{r_2}} \\
&\qquad \le c \left(\delta^2 + \delta^3 \right)^{\frac{r_2-2}{r_2}},
\end{aligned}
$$

where we have used Hölder inequality, Lemma 4.3.4, (4.2), (4.23), and (4.37). Then (4.38) implies

$$
\fint_{\Omega_4} |\nabla(u-w)|^2 + |p - p_w|^2 dx \le c \left(\delta^2 + \delta^{2-\frac{4}{r_2}} + \delta^{3-\frac{6}{r_2}} \right). \tag{4.39}
$$

According to (4.37) and (4.39), we discover that

$$
\fint_{\Omega_4} |\nabla w|^2 + |p_w|^2 dx \le c. \tag{4.40}
$$

Then we apply Lemma 4.3.6 to find that there exists a weak solution pair

(v, p_v) to (4.27) such that

$$\fint_{B_4^+} |\nabla v|^2 + |p_v|^2 dx \leq 1 \text{ and } \fint_{B_4^+} |w - v|^2 dx \leq \epsilon_*^2, \tag{4.41}$$

where ϵ_* is to be determined. We extend v from B_4^+ to B_4 by zero and then denote it by V. A direct computation and Lemma 4.2.8 imply that (V, p_V) is a weak solution pair to

$$\begin{cases} \text{div } (\bar{A}_{B_6^+} \nabla V) - \nabla p_V & = -\frac{\partial}{\partial x_n} \left(\bar{a}_{nn}^{\alpha\beta} \frac{\partial v^\alpha}{\partial x_n}(x', 0) \chi_{\mathbb{R}^n_-}(x) \right) & \text{in} & \Omega_4 \\ \text{div } V & = 0 & \text{in} & \Omega_4 \\ V & = 0 & \text{on} & \partial_w \Omega_4, \end{cases} \tag{4.42}$$

where $\bar{A}_{B_6^+} = \bar{a}_{ij}^{\alpha\beta}$, $v = (v^1, \cdots, v^n)$, $x' = (x_1, \cdots, x_{n-1})$ and χ is the standard characteristic function.

Note that $V \in W_\sigma^{1,2}(B_4)$ and $\nabla V = \nabla v$ a.e. in B_4^+, as $v = 0$ on T_4. Then it follows from Lemma 4.3.5 and (4.41) that

$$\|\nabla V\|_{L^\infty(\Omega_3)} = \|\nabla v\|_{L^\infty(B_3^+)} \leq c \|\nabla v\|_{L^2(B_4^+)} \leq c. \tag{4.43}$$

It follows from (4.26) and (4.42) that $(w - V, p_w - p_V)$ is a weak solution pair to

$$\begin{cases} -\text{div } (\bar{A}_{B_6^+} \nabla (w - V)) + \nabla (p_w - p_V) & = \left(\bar{a}_{nn}^{\alpha\beta} \frac{\partial v^\alpha}{\partial x_n}(x', 0) \chi_{\mathbb{R}^n_-}(x) \right)_{x_n} & \text{in} & \Omega_4 \\ \text{div } (w - V) & = 0 & \text{in} & \Omega_4 \\ w - V & = 0 & \text{on} & \partial_w \Omega_4. \end{cases}$$

Almost similarly as in the proof of Lemma 4.3.4, we obtain the following Caccioppoli type inequality

$$\fint_{\Omega_2} |\nabla (w - V)|^2 dx \leq c \left(\fint_{\Omega_3} |w - V|^2 dx + \fint_{\Omega_3} \left| \bar{a}_{nn}^{\alpha\beta} \frac{\partial v^\alpha}{\partial x_n}(x', 0) \chi_{\mathbb{R}^n_-}(x) \right|^2 dx \right). \tag{4.44}$$

The first term in right-hand side is estimated as follows.

$$\begin{aligned}
\fint_{\Omega_3} |w-V|^2 dx &\leq \fint_{B_3^+} |w-v|^2 dx + \frac{1}{|\Omega_3|}\int_{\Omega_3\setminus B_3^+} |w|^2 dx \\
&\leq \epsilon_*^2 + \frac{1}{|\Omega_3|}\left(\int_{\Omega_3\setminus B_3^+} |w|^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}} \left|\Omega_3 \setminus B_3^+\right|^{\frac{2}{n}} \\
&\leq \epsilon_*^2 + c\delta^{\frac{2}{n}} \fint_{\Omega_2} |\nabla w|^2 dx \\
&\leq \epsilon_*^2 + c\delta^{\frac{2}{n}},
\end{aligned}$$

where we have used Sobolev inequality, Hölder inequality, and (4.30). Using (4.2), (4.43) and (4.23), we estimate

$$\begin{aligned}
\fint_{\Omega_3} \left|\bar{a}_{nn}^{\alpha\beta}\frac{\partial v^\alpha}{\partial x_n}(x',0)\chi_{\mathbb{R}^n_-}(x)\right|^2 dx &\leq \frac{c}{|\Omega_3|}\int_{\Omega_3\setminus B_3^+} \left|\nabla v(x',0)\right|^2 dx \\
&\leq c\frac{|\Omega_3 \setminus B_3^+|}{|\Omega_3|} \\
&\leq c\delta.
\end{aligned}$$

Therefore, we deduce from (4.44) that

$$\fint_{\Omega_2} |\nabla(w-V)|^2 dx \leq \epsilon_*^2 + c(\delta + \delta^{\frac{2}{n}}). \tag{4.45}$$

Since an associated pressure is determined uniquely up to a constant, we assume $\fint_{\Omega_2} p_w - p_V \; dx = 0$. Then by Lemma 4.2.6, we can use inequality (0.8) in [43], which gives us

$$\begin{aligned}
&\|p_w - p_V\|_{L^2(\Omega_2)} \\
&\qquad \leq c\Big\|\mathrm{div}\;(\bar{A}_{B_6^+}\nabla(w-V)) - \frac{\partial}{\partial x_n}\left(\bar{a}_{nn}^{\alpha\beta}\frac{\partial v^\alpha}{\partial x_n}(x',0)\chi_{\mathbb{R}^n_-}(x)\right)\Big\|_{W^{-1,2}(\Omega_2)^n} \\
&\qquad \leq c\|\nabla k\|_{L^2(\Omega_2)^{n^2}},
\end{aligned}$$

where $k \in W_0^{1,2}(\Omega_2)^n$ satisfies the following weak formulation

$$\int_{\Omega_2} \nabla k : \nabla \phi \, dx = \int_{\Omega_2} \bar{A}_{B_6^+} \nabla (w - V) : \nabla \phi + \bar{a}_{nn}^{\alpha\beta} \frac{\partial v^\alpha}{\partial x_n}(x', 0) \chi_{\mathbb{R}^n_-}(x) \frac{\partial \phi^\beta}{\partial_{x_n}} \, dx$$

for all $\phi \in W_0^{1,2}(\Omega_2)^n$. Taking $\phi = k$, we have

$$\begin{aligned} \int_{\Omega_2} |\nabla k|^2 dx &\leq \int_{\Omega_2} \bar{A}_{B_6^+} \nabla (w - V) : \nabla k + \bar{a}_{nn}^{\alpha\beta} \frac{\partial v^\alpha}{\partial x_n}(x', 0) \chi_{\mathbb{R}^n_-}(x) \frac{\partial k^\beta}{\partial_{x_n}} \, dx \\ &\leq c \int_{\Omega_2} |\nabla (w - V)|^2 dx + \frac{1}{2} \int_{\Omega_2} |\nabla k|^2 dx + c\delta |\Omega_2|. \end{aligned}$$

Therefore, we have

$$\fint_{\Omega_2} |p_w - p_V|^2 dx \leq c\epsilon_*^2 + c(\delta + \delta^{\frac{2}{n}}). \tag{4.46}$$

Combining (4.45), (4.46) with (4.39) and taking ϵ_* and δ small enough, we complete the proof.

□

Lemma 4.3.8. *Given $F \in L^2(\Omega)^{n^2}$, let $(u, p) \in W_{0,\sigma}^{1,2}(\Omega)^n \times L^2(\Omega)$ be a weak solution pair to the steady Stokes system (4.1). Then there is a constant $N = N(\nu, L, n) > 0$ so that for any $\epsilon > 0$, there exists a small $\delta = \delta(\epsilon, \nu, L, n) > 0$ such that if A is $(\delta, 42)$-vanishing, Ω is $(\delta, 42)$-Reifenberg flat, and $B_r(y)$ for $r \in (0, 1]$ and $y \in \Omega$ satisfies*

$$\left| \{x \in \Omega : \mathcal{M}(|\nabla u|^2 + |p|^2)(x) \leq N^2\} \cap B_r(y) \right| \leq \epsilon |B_r(y)|, \tag{4.47}$$

then we have

$$\Omega_r(y) \subset \{x \in \Omega : \mathcal{M}(|\nabla u|^2 + |p|^2)(x) > 1\} \cup \{x \in \Omega : \mathcal{M}(|\mathbf{F}|^2)(x) > \delta^2\}. \tag{4.48}$$

Proof. We prove this lemma by contraposition. Assume that $B_r(y)$ satisfies (4.47) and that the conclusion (4.48) is false. Then there exists a point $y_1 \in \Omega_r(y)$ such that for all $\rho > 0$,

$$\fint_{\Omega_\rho(y_1)} |\nabla u|^2 + |p|^2 dx \leq 1, \quad \fint_{\Omega_\rho(y_1)} |\mathbf{F}|^2 dx \leq \delta^2. \tag{4.49}$$

We first consider the interior case that $B_{6r}(y) \subset \Omega$. Since $B_{5r}(y) \subset \Omega_{6r}(y_1)$, we see from (4.49) that

$$\fint_{B_{5r}(y)} |\nabla u|^2 + |p|^2 dx \leq \frac{|\Omega_{6r}(y_1)|}{|B_{5r}(y)|} \fint_{\Omega_{6r}(y_1)} |\nabla u|^2 + |p|^2 dx \leq \left(\frac{6}{5}\right)^n < 2^n. \tag{4.50}$$

In the same way, it follows that

$$\fint_{B_{5r}(y)} |\mathbf{F}|^2 dx \leq 2^n \delta^2. \tag{4.51}$$

We assume $y = 0$ and then consider the rescaled functions

$$\tilde{A}(x) = \frac{A(rx)}{2^{n/2}},\ (\tilde{u}(x), \tilde{p}(x)) = \left(\frac{u(rx)}{2^{n/2}r}, \frac{p(rx)}{2^{n/2}}\right),\ \tilde{\mathbf{F}}(x) = \frac{\mathbf{F}(rx)}{2^{n/2}},\ \text{and } \tilde{\Omega} = \frac{1}{r}\Omega.$$

With this setting, it is not difficult to see that all the assumptions of Lemma 4.3.3 are satisfied by Lemma 4.2.5, (4.50) and (4.51). Then according to Lemma 4.3.3 and Lemma 4.3.2, after scaling back, we find that there exists a pair $(v, p_v) \in W^{1,2}_\sigma(B^+_{3r}) \times L^2(B^+_{3r})$ such that

$$\|\nabla v\|_{L^\infty(B_{2r})^{n^2}} + \|p_v\|_{L^\infty(B_{2r})} \leq N_0 \text{ and } \fint_{B_{3r}} |\nabla(u-v)|^2 + |p - p_v|^2 dx \leq c_* \epsilon^2 \tag{4.52}$$

for some positive constant $N_0 = N_0(n, \nu, L)$, where c_* is to be determined in a universal way. We write $N_1 = max\{2N_0, 2^{n/2}\}$ to discover that

$$\{x \in B_r : \mathcal{M}(|\nabla u|^2 + |p|^2) > N_1^2\} \subset \{x \in B_r : \mathcal{M}_{B_{3r}}(|\nabla(u-v)|^2 + |p-p_v|^2) > N_0^2\}.$$

From this inclusion, Lemma 4.2.3 and (4.52), we conclude

$$\begin{aligned}
\frac{1}{|B_r|}&\big|\{x \in B_r : \mathcal{M}\big(|\nabla u|^2 + |p|^2\big)(x) > N_1^2\}\big| \\
&\leq \frac{1}{|B_r|} \left|\left\{x \in B_r : \mathcal{M}_{B_{4r}} \left(|\nabla(u-v)|^2 + |p-p_v|^2\right)(x) > N_1^2\right\}\right| \\
&\leq c \fint_{B_r} |\nabla(u-v)|^2 + |p-p_v|^2 dx \\
&\leq c c_* \epsilon^2 < \epsilon,
\end{aligned}$$

by taking sufficiently small c_* in order to have the last inequality. This is a contradiction to (4.47).

We next consider the boundary case that $B_{6r}(y) \not\subset \Omega$. In this case, there is a boundary point $y_0 \in \partial\Omega \cap B_{6r}(y)$. By the Reifenberg flatness condition and the small BMO condition, we assume that there exists a new coordinate system through suitable orientation and translation, depending on y_0 and r, so that in this new coordinate system, the origin is $y_0 + \delta_0 \overrightarrow{n_0}$ for some small $\delta_0 > 0$ and for some inward unit normal $\overrightarrow{n_0}$. We now denote the variable as z in the new coordinate, $y_0 = z_0$ and $y_1 = z_1$. The we find

$$B_{42r}^+ \subset \Omega_{42r} \subset \{z \in B_{42r} : z_n > -84r\delta\} \tag{4.53}$$

and

$$\fint_{\Omega_{42r}} \left|A(z) - \overline{A}_{\Omega_{42r}}\right|^2 dz \leq \delta^2. \tag{4.54}$$

Moreover, it follows from (4.49) that

$$\fint_{\Omega_{35r}} |\nabla u|^2 + |p|^2 dx \leq \frac{|B_{42r}|}{|B_{35r}^+|} \fint_{\Omega_{42r}} |\nabla u|^2 + |p|^2 dx \leq 2\left(\frac{6}{5}\right)^n < 2^{n+1} \tag{4.55}$$

and

$$\fint_{\Omega_{35r}} |\mathbf{F}|^2 dx \leq 2^{n+1}\delta^2. \tag{4.56}$$

We apply Lemma 4.2.5 to $\rho = 7r$ and $\lambda = 2^{\frac{n+1}{2}}$ to see that all the assumptions of Lemma 4.3.7 are satisfied by (4.53), (4.54) and (4.55). As a consequence, we find that there exists a function $V \in W_\sigma^{1,2}(\Omega_{28r})$ such that

$$\|\nabla V\|_{L^\infty(\Omega_{21r})^{n^2}} + \|p_V\|_{L^\infty(\Omega_{21r})} \leq N_2$$

for some constant $N_2 = N_2(\nu, L, n)$ and

$$\fint_{\Omega_{7r}} |\nabla(u - V)|^2 + |p - p_V|^2 dx \leq c_1\epsilon,$$

where c_1 is to be determined.

As in the interior case, putting $N_3 = \max\{2N_2, 2^{\frac{n}{2}}\}$, we conclude

$$\frac{1}{|B_{7r}|}\left|\{z \in \Omega : (|\nabla u|^2 + |p|^2) > N_3^2\} \cap B_{7r}\right| \leq cc_1\epsilon,$$

which implies that

$$\frac{1}{|B_r|}\left|\{z \in \Omega : (|\nabla u|^2 + |p|^2) > N_3^2\} \cap B_r\right| \leq cc_1\epsilon < \epsilon,$$

by taking c_1 so that the last inequality holds. Finally, we set $N = \max\{N_1, N_3\}$ to complete the proof. □

Lemma 4.3.9. *Assume that $\omega \in A_s$ for some $s \in (1, \infty)$. Given $F \in L^2_\omega(\Omega)^n$, let $(u, p) \in W^{1,2}_{0,\sigma}(\Omega)^n \times L^2(\Omega)$ be a weak solution pair to the steady Stokes system (4.1). Then there is a constant $N = N(\nu, L, n) > 0$ so that for any $\epsilon > 0$ there exists a small $\delta = \delta(\epsilon, \nu, L, q, \omega) > 0$ such that if A is $(\delta, 42)$-vanishing, Ω is $(\delta, 42)$-Reifenberg flat, and $B_r(y)$ for $r \in (0, 1]$ and $y \in \Omega$ satisfies*

$$\omega\left(\{x \in \Omega : \mathcal{M}(|\nabla u|^2 + |p|^2)(x) \leq N^2\} \cap B_r(y)\right) \leq \epsilon\omega\left(B_r(y)\right), \tag{4.57}$$

then we have

$$\Omega_r(y) \subset \{x \in \Omega : \mathcal{M}(|\nabla u|^2 + |p|^2)(x) > 1\} \cup \{x \in \Omega : \mathcal{M}(|\mathbf{F}|^2)(x) > \delta^2\}. \tag{4.58}$$

Proof. From Lemma 4.2.1 and (4.57), we have

$$\begin{aligned}
&\left|\{x \in \Omega : \mathcal{M}(|\nabla u|^2 + |p|^2) > N^2\} \cap B_r(y)\right| \\
&\qquad \geq \left(\frac{1}{\mu}\frac{\omega(\{x \in \Omega : \mathcal{M}(|\nabla u|^2 + |p|^2) > N^2\} \cap B_r(y))}{\omega(B_r(y))}\right)^{\frac{1}{\tau}} |B_r(y)| \\
&\qquad \geq \left(\frac{\epsilon}{\mu}\right)^{\frac{1}{\tau}} |B_r(y)|.
\end{aligned}$$

We use Lemma 4.3.8 with ϵ replaced by $\left(\frac{\epsilon}{\mu}\right)^{\frac{1}{\tau}}$, to find $\delta = \delta(\epsilon, \nu, L, n, \omega, s)$ so that (4.58) holds. □

We are now all set to prove the main result.

Proof of Theorem 4.1.4. We first assert that

$$\|\nabla u\|_{L^q_\omega(\Omega)^{n^2}} + \|p\|_{L^q_\omega(\Omega)} \leq c, \;\text{ if } \|\mathbf{F}\|_{L^q_\omega(\Omega)^{n^2}} \leq \delta \tag{4.59}$$

for some constant $c = c(n, q, \nu, L, \Omega, \omega)$. To do this, we compute

$$\|\mathbf{F}\|^2_{L^2(\Omega)^{n^2}} = \int_\Omega |\mathbf{F}|^2 \omega^{\frac{2}{q}} \omega^{-\frac{2}{q}} dx \le \left(\int_\Omega |\mathbf{F}|^q \omega \, dx\right)^{\frac{2}{q}} \left(\int_\Omega \omega^{\frac{-2}{q-2}} dx\right)^{\frac{q-2}{q}}. \quad (4.60)$$

Since Ω is bounded, there is a ball $B_{\frac{d}{2}}(x_0) \supset \Omega$ for some $x_0 \in \Omega$, where d is the diameter of Ω. Using (4.5) and (4.6), we estimate

$$\begin{aligned}
\left(\int_\Omega \omega^{\frac{-2}{q-2}} dx\right)^{\frac{q-2}{2}} &\le \left(\int_{B_{\frac{d}{2}}(x_0)} \omega^{\frac{-2}{q-2}} dx\right)^{\frac{q-2}{2}} \\
&= \left(\fint_{B_{\frac{d}{2}}(x_0)} \omega \, dx\right) \left(\fint_{B_{\frac{d}{2}}(x_0)} \omega \, dx\right)^{-1} \left(\fint_{B_{\frac{d}{2}}(x_0)} \omega^{\frac{-2}{q-2}} dx\right)^{\frac{q}{2}-1} \left|B_{\frac{d}{2}}\right|^{\frac{q}{2}-1} \\
&\le \frac{\left|B_{\frac{d}{2}}\right|^{\frac{q}{2}}}{\omega\left(B_{\frac{d}{2}}(x_0)\right)} [\omega]_{\frac{q}{2}} \le \frac{d^{\frac{nq}{2}} |B_1|^{\frac{q}{2}}}{\omega(\Omega)} [\omega]_{\frac{q}{2}}.
\end{aligned}$$

Thus from (4.59) and (4.60), we have

$$\|\mathbf{F}\|^2_{L^2(\Omega)^{n^2}} \le \frac{d^n |B_1|}{\omega(\Omega)^{\frac{2}{q}}} [\omega]^{\frac{2}{q}}_{\frac{q}{2}} \delta^2. \quad (4.61)$$

We now take $\epsilon \in (0,1)$ and N and choose the corresponding δ given by Lemma 4.3.9. Then write

$$\begin{aligned}
\mathfrak{C} &= \left\{x \in \Omega : \mathcal{M}\left(|\nabla u|^2 + |p|^2\right)(x) > N^2\right\} \text{ and} \\
\mathfrak{D} &= \left\{x \in \Omega : \mathcal{M}\left(|\nabla u|^2 + |p|^2\right)(x) > 1\right\} \cup \left\{x \in \Omega : \mathcal{M}\left(|\mathbf{F}|^2\right)(x) > \delta^2\right\}.
\end{aligned}$$

By using Lemma 4.2.3, (4.9) and (4.61), one can check that the first hypothesis of Lemma 4.2.4 as follows.

$$\begin{aligned}|\mathfrak{C}\cap B_1(y)| &\leq c\int_\Omega |\nabla u|^2+|p|^2dx\\ &\leq c\int_\Omega |\mathbf{F}|^2dx\\ &\leq c\delta^2\\ &\leq \left(\frac{\epsilon}{\mu}\right)^{\frac{1}{\tau}}|B_1|,\end{aligned}$$

by choosing a small enough δ, if necessary, in order to get the last inequality. Then Lemma 4.2.1 implies

$$\omega\big(\mathfrak{C}\cap B_1(y)\big)\leq \mu\left(\frac{|\mathfrak{C}\cap B_1(y)|}{|B_1|}\right)^\tau \omega\big(B_1(y)\big)\leq \epsilon\omega\big(B_1(y)\big).$$

On the other hand, the second hypothesis of Lemma 4.2.4 follows directly from Lemma 4.3.9. Therefore thanks to Lemma 4.2.4, we have

$$\begin{aligned}&\omega\left(\left\{x\in\Omega:\mathcal{M}\left(|\nabla u|^2+|p|^2\right)(x)>N^2\right\}\right)\\ &\quad\leq \epsilon_1\omega\left(\left\{x\in\Omega:\mathcal{M}\left(|\nabla u|^2+|p|^2\right)(x)>1\right\}\right)+\epsilon_1\omega\left(\left\{x\in\Omega:\mathcal{M}\left(|\mathbf{F}|^2\right)(x)>\delta^2\right\}\right)\end{aligned} \tag{4.62}$$

for $\epsilon_1=c^*\epsilon$, where c^* depends only on $n,q,[\omega]_{\frac{q}{2}}$. Using an iteration argument from (4.62), we further have the following power decay estimate.

$$\begin{aligned}\omega\Bigg(\{x\in\Omega:\mathcal{M}(|\nabla u|^2+|p|^2)(x)>N^{2k}\}\Bigg)&\\ \leq \epsilon_1^k\omega\Big(\left\{x\in\Omega:\mathcal{M}\left(|\nabla u|^2+|p|^2\right)(x)>1\right\}\Big)& \quad (4.63)\\ +\sum_{i=1}^k\epsilon_1^i\omega\left(\left\{x\in\Omega:\mathcal{M}\left(|\mathbf{F}|^2\right)(x)>\delta^2N^{(k-i)2}\right\}\right).&\end{aligned}$$

Then using this estimate (4.63), we estimate

$$\begin{aligned}
&\sum_{k=1}^{\infty} N^{qk}\omega\bigg(\left\{x\in\Omega : \mathcal{M}(|\nabla u|^2+|p|^2)(x)>N^{2k}\right\}\bigg)\\
&\qquad\le \sum_{k=1}^{\infty}(N^q\epsilon_1)^k\omega\left(\left\{x\in\Omega : \mathcal{M}\left(|\nabla u|^2+|p|^2\right)(x)>1\right\}\right)\\
&\qquad\quad+\sum_{i=1}^{\infty}(N^q\epsilon_1)^i\underbrace{\sum_{k=i}^{\infty}N^{q(k-i)}\omega\left(\left\{x\in\Omega : \mathcal{M}\left(|\mathbf{F}|^2\right)(x)>\delta^2N^{(k-i)2}\right\}\right)}_{S}\\
&\qquad\le \sum_{k=1}^{\infty}(N^q\epsilon_1)^k\omega(\Omega)+\sum_{i=1}^{\infty}(N^q\epsilon_1)^i\ S.
\end{aligned}$$

We next show that S is finite. In light of Lemma 4.2.2, Lemma 4.2.3 and the assumption that $\|\mathbf{F}\|_{L^q_\omega(\Omega)^{n^2}}\le\delta$, we compute

$$S\le c\frac{1}{\delta}\|\mathbf{F}\|_{L^q_\omega(\Omega)^{n^2}}\le c$$

for some $c=c(n,q,\nu,L,\omega,\Omega)$. Consequently, we discover

$$\sum_{k=1}^{\infty}N^{qk}\omega\left(\left\{x\in\Omega : \mathcal{M}\left(|\nabla u|^2+|p|^2\right)(x)>N^{2k}\right\}\right)\le c\sum_{k=1}^{\infty}(N^q\epsilon_1)^k\le c,$$

by selecting ϵ so small that $N^q\epsilon_1<1$. Therefore, the assertion (4.59) is now proved by Lemma 4.2.2.

To derive the desired estimate (4.7) in Theorem 4.1.4, we consider the normalized functions as

$$u_\lambda=\frac{u}{\lambda},\ p_\lambda=\frac{p}{\lambda}\ \text{and}\ \mathbf{F}_\lambda=\frac{\mathbf{F}}{\lambda},$$

where $\lambda=\delta^{-1}\|\mathbf{F}\|_{L^q_\omega(\Omega)^{n^2}}$. Then it follows that

$$\|\mathbf{F}_\lambda\|_{L^q_\omega(\Omega)^{n^2}}\le\delta.$$

Then (4.59) implies that there is a constant $c = c(n, q, \nu, L, \omega, \Omega)$ such that

$$\|\nabla u_\lambda\|_{L^q_\omega(\Omega)^{n^2}} + \|p_\lambda\|_{L^q_\omega(\Omega)} \leq c,$$

which is (4.7). This completes the proof.

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국문초록

이 논문에서는 볼록영역에서의 G-라플라스 방정식, 준볼록영역에서의 준선형 방정식, 라이펜버그 평탄 영역에서의 일반화된 스트크스 방정식 세가지 문제를 다룬다. 각 문제에서 연구주제는 약해의 그래디언트 가늠이다.

첫번째 문제에서는 볼록영역에서 정의된 영이 되는 노이만 경계 조건을 갖는 G-라플라스 동형방정식의 국소적 립쉬츠 경계 정칙성을 증명한다. G는 $p \in (1, \infty)$를 만족할 때 t^p를 일반화시킨 영함수이다. 이 문제에서 핵심사항은 볼록영역이다. 왜냐하면 립쉬츠 영역에서는 립쉬츠 정칙성을 얻을 수 없기 때문이다.

그 다음 장에서는 준볼록영역에서 정의된 p성장 조건을 갖는 준선형 방정식의 칼데론-지그문트 형태의 가늠을 증명한다. 여기서 준볼록영역은 경계를 국소적으로 두 개의 볼록영역으로 가둘 수 있는 영역이다. 방정식이 정의된 영역의 정칙성의 관점에서 이 논문에서 가정한 경계의 정칙성은 이 주제에서 지금까지 알려진 가장 약한 조건이다. 추가로 르벡공간에서 오리쯔 공간으로 결과를 확장하였다.

마지막 장에서는 충분히 평탄한 라이펜버그 영역에서 정의된 작은 유계평균진동 반노름을 갖는 계수를 가진 일반화된 스토크스 방정식의 약해의 그래디언트의 대역적 가중L^q가늠을 증명하였다. 주어진 가중값은 머켄하우프트 류에 속한다고 가정하였다. 이 결과는 립쉬츠 영역에서 정의된 스토크스 방정식의 르벡측도에 대한 대역적 칼데론-지그문트 $W^{1,q}$ 가늠을 일반화하였다.

주요어휘: 준선형 타원형 방정식, 립쉬츠 연속, 대역적 그래디언트 가늠, 스토크스 방정식, 비정칙 영역.

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